SYMPLECTIC REPRESENTATIONS OF BRAID GROUPS

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1 Braid Groups and Mapping Class Groups

Let X be a Riemann surface (for our purposes, X will always be either the complex plane or a compact Riemann surface). We will denote by $F_n(X)$ the *ordered configuration space* of n points on X, which is given by X^n minus all subsets $\Delta_{i,j}$ of points $(z_1, ..., z_n)$ with $z_i = z_j$, for $1 \leq i, j \leq n$. In other words, $F_n(X)$ is the spaces of ordered subsets of X of cardinality n. The symmetric group S_n acts on $F_n(X)$ by $\sigma \cdot (z_1, ...z_n) = (z_{\sigma(1)}, ..., z_{\sigma(n)})$ for all permutations $\sigma \in S_n$. We will denote by $C_n(X)$ the *(unordered) configuration space* of n points on X, which is given by taking the quotient of $F_n(X)$ by this action of S_n. One can think of $C_n(X)$ as the space of (unordered) subsets of of X of cardinality n.

The fundamental group of $C_n(X)$ is the *(full) braid group* on n strands of X and will be denoted $B_n(X)$. The fundamental group of $F_n(X)$ can be identified with all braids in $B_n(X)$ which return the n points to their original positions (rather than merely permute them). It is called the *pure braid group* on n strands of X and will be denoted $P_n(X)$. Note that, as one expects from covering space theory, $P_n(X) \lhd B_n(X)$, and $B_n(X)/P_n(X) \cong S_n$. In fact, the above isomorphism can be defined using the obvious homomorphism $B_n(X) \to S_n$, which takes each braid to its corresponding permuation of n points, and whose kernel is $P_n(X)$.

Let P denote the complex plane, and let let T_0 denote the Riemann sphere. The following presentations of their braid groups are computed in [1].

Fact 1.1. *a) The braid group on n strands of P, often denoted simply by B_n, is generated by elements $\beta_1, ..., \beta_{n-1}$, where each β_i is the braid switching the ith and $(i+1)$th strands by pulling the ith strand over the $(i+1)$th strand. A full set of relations is*

$$\beta_i\beta_j = \beta_j\beta_i, \ |i-j| \geq 2,$$

$$\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}, \ 1 \leq i \leq n-2.$$

b) The pure braid group on n strands on P, often denoted simply by P_n, is generated by all braids of the form

$$A_{i,j} = \beta_{j-1}\beta_{j-2}...\beta_{i+1}\beta_i^2\beta_{i+1}^{-1}...\beta_{j-1}^{-1}\beta_j^{-1}$$

for $1 \leq i < j \leq n$. Each $A_{i,j}$ wraps the jth strand around the ith strand by going under and then over and back to its original position. (Relations omitted.)

c) For $n \geq 3$, B_n has infinite cyclic center generated by

$$(\beta_1\beta_2...\beta_{n-1})^n = (A_{1,2})(A_{1,3}A_{2,3})...(A_{1,n}A_{2,n}...A_{n-1,n}).$$

Fact 1.2. *a) The braid group on n strands of T_0 is the quotient of B_n by the relation*

$$(\beta_1...\beta_{n-2}\beta_{n-1})(\beta_{n-1}\beta_{n-2}...\beta_1) = 1.$$

b) Identifying each $\beta_i \in B_n$ with its image in $B_n(T_0)$, the center of $B_n(T_0)$ is generated by $(\beta_1\beta_2...\beta_{n-1})^n$, which is of order 2 in $B_n(T_0)$.

The structure of the braid groups of higher genus Riemann surfaces is more complicated but can be determined using certain short exact sequences and induction on n.

Let T_g be the compact Riemann surface of genus $g \geq 0$, and let $\{z_1, ..., z_n\}$ be subset of T_g of cardinality n. Consider the set $\mathcal{C}_n(T_g)$ of orientation-preserving homeomorphisms of T_g to itself which fix the subset $\{z_1, ..., z_n\}$ (thus permuting the z_i's), which forms a group under composition. Let two self-homeomorphisms $f, g \in \mathcal{C}_n(T_g)$ be "equivalent" if there is a continuous transformation from f to g. Let $M_n(T_g)$ denote the *(full) mapping class group* of T_g, which is $\mathcal{C}_n(T_g)$ modulo the equivalence relation described above. We may also define the subgroup $\mathcal{F}_n(T_g)$ of $\mathcal{C}_n(T_g)$ consisting of the self-homeomorphisms which fix each point z_i, and the corresponding quotient group $M_n^0(T_g)$, called the *pure mapping class group* of T_g.

There is clearly an inclusion $\iota_{g,n} : \mathcal{C}_n(T_g) \hookrightarrow \mathcal{C}_0(T_g)$, and an analogous inclusion $\mathcal{F}_n(T_g) \hookrightarrow \mathcal{F}_0(T_g)$ which, by abuse of notation, we will also denote $\iota_{g,n}$. Now define the "evaluation map" $\epsilon_{g,n} : \mathcal{F}_0(T_g) \to F_n(T_g)$ (respectively, again by abuse of notation, $\epsilon_{g,n} : \mathcal{C}_0(T_g) \to C_n(T_g)$) as follows: For any self-homeomorphism f in $\mathcal{F}_0(T_g)$ (respectively $\mathcal{C}_n(T_g)$), $\epsilon_{g,n}(f)$ is $(f(z_1), ..., f(z_n)) \in F_n(T_g)$ (respectively the image of $(f(z_1), ..., f(z_n))$ in $C_n(T_g)$). The following lemma, proven in [1], connects $\iota_{g,n}$ with $\epsilon_{g,n}$.

Lemma 1.3. *The map $\epsilon_{g,n} : \mathcal{F}_0(T_g) \to F_n(T_g)$, considered as a continuous map from the space $\mathcal{F}_0(T_g)$ with the compact-open topology to the space $F_n(T_g)$ with the obvious topology, is a locally trivial fibering with fiber $\mathcal{F}_n(T_g)$.*

Now for any $n \geq 0$, considering $\mathcal{F}_0(T_g)$ as a topological space under the compact-open topology, $M_n^0(T_g)$ is its group of connected components. It follows that there is an induced homomorphism $(\iota_{g,n})_* : M_n^0(T_g) \to M_0^0(T_g)$ (and similarly, we can define $(\iota_{g,n})_* : M_n(T_g) \to M_0(T_g)$). In fact, the fibering $\mathcal{F}_n(T_g) \to \mathcal{F}_0(T_g) \to F_n(T_g)$ induces a long exact sequence of fundamental groups (similarly for $\mathcal{C}_n(T_g) \to \mathcal{C}_0(T_g) \to C_n(T_g)$). Keeping in mind that $F_n(T_g)$ (respectively $C_n(T_g)$) has one connected component, and that $P_n(T_g) = \pi_1(F_n(T_g), (z_1, ..., z_n))$ (respectively, $B_n(T_g) = \pi_1(C_n(T_g), \{z_1, ..., z_n\})$), we get long exact sequences as follows.

Theorem 1.4. *There is a long exact sequence of groups ending in*

$$... \to \pi_1(\mathcal{F}_0(T_g)) \to P_n(T_g) \to M_n^0(T_g) \to M_0^0(T_g) \to 1$$

and, similarly, a long exact sequence of groups ending in

$$... \to \pi_1(\mathcal{C}_0(T_g)) \to B_n(T_g) \to M_n(T_g) \to M_0(T_g) \to 1.$$

The map $(\iota_{g,n})_* : M_n^0(T_g) \to M_0^0(T_g)$ (respectively $(\iota_{g,n})_* : M_n(T_g) \to M_0(T_g)$) is induced by $\iota_{g,n}$, and the map $(\epsilon_{g,n})_* : \pi_1(\mathcal{F}_0(T_g)) \to P_n(T_g)$ (respectively $(\epsilon_{g,n})_* : \pi_1(\mathcal{C}_0(T_g) \to B_n(T_g))$ is induced by $\epsilon_{g,n}$.

The map $\delta_{g,n} : P_n(T_g) \to M_n^0(T_g)$ (similarly, $\delta_{g,n} : B_n(T_g) \to M_n(T_g)$ has a particularly interesting construction, which will be described on the pure braid group $P_n(T_g)$; the extension to $B_n(T_g)$ will be obvious. Namely, let β be a braid in $P_n(T_g)$, considered as a continuous map $\beta : [0,1] \to F_n(T_g)$ such that $\beta(0) = \beta_1 = (z_1, ..., z_n)$. It is easy to construct a homotopy of self-homeomorphisms of T_g, call it $H : T_g \times [0,1] \to T_g$, such that $H(\cdot, 0)$ is the identity map on T_g, and for $1 \leq i \leq n$ and $s \in [0,1]$, $H(z_i, s)$ is the ith projection of $\beta(s)$. Then $\delta_{g,n}(\beta) = H(\cdot, 1)$. It is easy to see that, as we already know from the exact sequence in Theorem 1.4, the image of $\delta_{g,n}$ equals the kernel of $(\iota_{g,n})_*$. This amounts to noting that for any $\beta \in P_n(T_g)$, the homotopy H constructed for δ as above is a transformation from $1 \in M_0^0(T_g)$ to $\delta_{g,n}(\beta) \in M_0^0(T_g)$, and that conversely, for an equivalence between 1 and any $f \in M_0^0(T_g)$ which fixes each z_i, the motion of the points z_i under this homotopy constitutes a braid.

Theorem 1.5. *The kernel of $(\iota_{g,n})_*$ (and hence, the image of $\delta_{g,n}$) is isomorphic to $P_n(T_g)/Z(P_n(T_g))$ for $g \geq 2$, $g = 1$ and $n \geq 2$, and $g = 0$ and $n \geq 3$. In particular, for $g \geq 2$, $P_n(T_g)$ has trivial center and therefore, $\ker((\iota_{g,n})_*) = \mathrm{im}(\delta_{g,n}) \cong P_n(T_g)$. The same holds in the analogous situation with $B_n(T_g)$.*

Proof. This is proven in [1], first by showing that $\ker(\delta_{g,n}) = Z(P_n(T_g))$, and then by calculating the centers of (almost) all $P_n(T_g)$'s. Note for the situation with $B_n(T_g)$, that $Z(B_n(T_g)) \leq Z(P_n(T_g))$.

□

Corollary 1.6. *Let $n \geq 2$ be given. The map $\delta_{0,n} : P_n(T_0) \to M_n^0(T_0)$ is a surjection whose kernel is the center of $P_n(T_0)$. Similarly, the map $\delta_{0,n} : B_n(T_0) \to M_n(T_0)$ is a surjection whose kernel is the center of $B_n(T_0)$.*

Proof. This comes directly from the exact sequence of Theorem 1.4, the statement of Theorem 1.5, and the fact that $M_0^0(T_0) = M_0(T_0) = 1$, which comes a "well-known folk theorem" that is proven in [1].

□

Corollary 1.7. *For $n \geq 2$, $M_n(T_0)$ is generated by elements $\omega_1, ..., \omega_{n-1}$ with the following relations:*

$$\omega_i\omega_j = \omega_j\omega_i, \ |i-j| \geq 2,$$

$$\omega_i\omega_{i+1}\omega_i = \omega_{i+1}\omega_i\omega_{i+1}, \ 1 \leq i \leq n-1,$$

$$(\omega_1...\omega_{n-1})(\omega_{n-1}...\omega_1) = 1,$$

$$(\omega_1...\omega_{n-1})^n = 1.$$

Proof. This results from Corollary 1.6, letting $\omega_i = \delta_{0,n}(\beta_i)$. The first three relations are the relations of the β_i's in $B_n(T_0)$ given in Facts 1.1 and 1.2, and the last one is the generator of the center of $B_n(T_0)$, by 1.2(b).

□

2 Application to Families of Hyperelliptic Curves

In this section, we explore monodromy representations of certain braid groups and apply results from the last section to determine the kernels of these representations. Recall that a hyperelliptic curve of genus $g \geq 1$ can be

characterized as a two-sheeted cover of the Riemann sphere T_0 ramified at exactly $2g+2$ points, and complex analytically can be viewed as a g-holed torus. (For $g = 1$, this is simply an elliptic curve.) Moreover, any choice of $2g+2$ distinct points on T_0 uniquely determines a hyperelliptic curve. Thus, we may construct a family of hyperelliptic curves over the (unordered) configuration space $C_{2g+2}(T_0)$ of (unordered) subsets of $2g+2$ points on T_0. For simplicity, we will restrict this family by requiring the $2g+2$th point to be fixed at $\infty \in T_0$. Thus, the base of the family can be viewed as $C_{2g+1}(P)$.

To be more precise, let $E_g \to C_{2g+1}(P)$ be the family of hyperelliptic curves such that for each set $Z \in C_{2g+1}(P)$, the fiber $(E_g)_Z$ is the hyperelliptic curve branched over T_0 at ∞ and the $2g+1$ points in Z. Meanwhile, let $E'_g \to C_{2g+1}(P)$ be the family of punctured hyperelliptic curves such that for each set $Z \in C_{2g+1}(P)$, the fiber $(E'_g)_Z$ is the hyperelliptic curve $(E_g)_Z$ minus its $2g+2$ branch points. Denote this set of branch points by B_Z, or just B when it is understood that a certain Z has been fixed. Thus, $(E'_g)_Z$ is a covering space of degree 2 of $P \backslash Z$.

Because each $(E'_g)_Z$ is a covering space of $P \backslash Z$, it is easier to do calculations on the family E'_g. Since $E'_g \to C_{2g+1}(P)$ is a fibering whose fibers are copies of T_g minus $2g+2$ points, and these are all "nice" spaces, if we choose a basepoint $Z = \{z_1, ..., z_{2g+1}\} \in C_{2g+1}(P)$ and let $B = B_Z$, we get a short exact sequence of fundamental groups

$$1 \to \pi_1(T_g \backslash B) \to \pi_1(E'_g) \to \pi_1(C_{2g+1}(P)) \to 1.$$

A similar short exact sequence can be constructed for the family $E_g \to C_{2g+1}(P)$, whose fibers are copies of T_g.

It is not difficult to find a continuous section $s : C_{2g+1}(P) \to E'_{2g+1}$, which induces a section $s_* : \pi_1(C_{2g+1}(P)) \to \pi_1(E'_g)$ making the above short exact sequence split. This induces an action of $\pi_1(C_{2g+1}(P)) \cong B_{2g+1}$ on $\pi_1(T_g \backslash B)$, by letting $s_*(\beta) \in \pi_1(E'_g)$ act on $\pi_1(T_g \backslash B) \lhd \pi_1(E'_g)$ by conjugation, for any $\beta \in \pi_1(C_{2g+1}(P))$. Similarly, for the family E_g, one defines an action of $\pi_1(C_{2g+1}(P)) \cong B_{2g+1}$ on $\pi_1(T_g)$ in exactly the same way. This action is determined in the statement of the following proposition.

Proposition 2.1. *Let $\beta_1, ..., \beta_{2g}$ be the generators of $\pi_1(C_{2g+1}(P)) \cong B_{2g+1}$ with the relations given in Fact 1.1. There is a set of generators $\alpha_1, ..., \alpha_{2g}$ of $\pi_1(T_g)$ with the single relation $[\alpha_1, \alpha_2][\alpha_3, \alpha_4]...[\alpha_{2g-1}, \alpha_{2g}] = 1$, such that the aforementioned action of $\pi_1(C_{2g+1}(P))$ on $\pi_1(T_g)$ is given by*

$$\begin{aligned}\beta_i \cdot \alpha_{i-1} &= \alpha_{i-1}\alpha_i,\\ \beta_i \cdot \alpha_i &= \alpha_i,\\ \beta_i \cdot \alpha_{i+1} &= \alpha_i^{-1}\alpha_{i+1},\\ \beta_i \cdot \alpha_j &= \alpha_j,\ j \neq i-1, i, i+1.\end{aligned}$$

Proof. First, note that $E_g'\backslash B$ is a covering space of degree 2 of $P\backslash\{z_1, ..., z_{2g+1}\}$. Fixing a basepoint z_{2g+2} in P different from $z_1, ..., z_{2g+1}$, it is well known that the fundamental group of $P\backslash\{z_1, ..., z_{2g+1}\}$ is generated by loops x_i of z_{2g+2} traveling around each z_i for $1 \leq i \leq 2g+1$, with the single relation $x_1x_2...x_{2g+2} = 1$. Therefore, $\pi_1(P\backslash Z)$ is the free group F_{2g+1} on the $2g+1$ generators $x_1, ..., x_{2g+1}$. If we fix a basepoint of $T_g\backslash B$ lying over z_{2g+2}, the loops in $\pi_1(P\backslash Z)$ which lift to closed loops of T_g correspond to exactly those words of F_{2g+1} with even exponent. The (normal) subgroup of words in F_{2g+1} with even exponent is easily seen to be freely generated by the words $x_1^2, x_1x_2, x_2^2, x_2x_3, ..., x_{2g}x_{2g+1}, x_{2g+1}^2$. Thus, $\pi_1(P\backslash B)$ can be presented as a free group on the above set of $4g+1$ generators.

Meanwhile, $\pi_1(T_g)$ can be obtained from $\pi_1(T_g\backslash B)$ by making the the loops around each branch point in B trivial. Each branch point in B lies above a point $z_i \in Z$, and a loop around such a branch point is the lifting of the double loop x_i^2 around z_i. Since $x_{2g+2} = (x_1x_2...x_{2g+1})^{-1}$, it follows that $\pi_1(T_g)$ is simply $\pi_1(T_g\backslash B)$ modulo the relations $x_1^2 = ... = x_{2g+1}^2 = (x_1x_2...x_{2g+1})^{-2} = 1$. Letting α_i be the image of x_ix_{i+1} for $1 \leq i \leq 2g$, once can calculate that the α_i's generate $\pi_1(T_g)$ with the single relation $[\alpha_1, \alpha_2][\alpha_3, \alpha_4]...[\alpha_{2g-1}, \alpha_{2g}] = 1$.

To calculate the action of $\pi_1(C_{2g+2}(T_g))$ on each $x_i \in \pi_1(T_0\backslash Z)$ for $1 \leq i \leq 2g+1$, one may consider x_i to be a braid on the $2g+2$ strands $z_1, ..., z_{2g+1}, z_{2g+2}$ (in other words, the basepoint is also a strand, which is being wrapped once around the strand z_i). Letting β_{2g+1} denote the braid switching z_{2g+1} and z_{2g+2} by wrapping the strand z_{2g+1} over the strand z_{2g+2}, we may write

$$x_i = \beta_{2g+1}\beta_{2g}...\beta_{i+1}\beta_i^2\beta_{i+1}^{-1}...\beta_{2g}^{-1}\beta_{2g+1}^{-1}.$$

Then one can calculate

$$\begin{aligned}\beta_i \cdot x_i &= \beta_i x_i \beta_i^{-1} = x_{i+1},\\ \beta_i \cdot x_{i+1} &= \beta_i x_{i+1} \beta_i^{-1} = x_{i+1}^{-1} x_i x_{i+1},\\ \beta_i \cdot x_j &= \beta_i x_j \beta_i^{-1} = x_j,\ j \neq i, i+1.\end{aligned}$$

Restricting this action to the generators of $\pi_1(T_g \backslash B)$ and then taking the quotient modulo the relations $x_1^2 = ... = x_{2g+1}^2 = (x_1 x_2 ... x_{2g+1})^{-2} = 1$ and letting α_i be the image of $x_i x_{i+1}$ for $1 \leq i \leq 2g$, we get the statement of the proposition.

□

Corollary 2.2. *The homology group $H_1(T_g, \mathbb{Z})$ can be generated as a free $\mathbb{Z}$-module by the images of the generators $\alpha_1, ..., \alpha_{2g}$ of $\pi_1(T_g)$ (which by abuse of notation will also be denoted $\alpha_1, ..., \alpha_{2g}$). The induced action of $\pi_1(C_{2g+2}(P))$ on $H_1(T_g, \mathbb{Z})$ is given by*

$$\begin{aligned} \beta_i \cdot \alpha_{i-1} &= \alpha_{i-1} + \alpha_i, \\ \beta_i \cdot \alpha_i &= \alpha_i, \\ \beta_i \cdot \alpha_{i+1} &= \alpha_{i+1} - \alpha_i, \\ \beta_i \cdot \alpha_j &= \alpha_j, \ j \neq i-1, i, i+1. \end{aligned}$$

Proof. This immediately follows from Proposition 2.1 by noting that $H_1(T_g, \mathbb{Z})$ is just the abelianization of $\pi_1(T_g)$.

□

The loops $\alpha_1, ..., \alpha_{2g}$ which generate the homology group $H_1(T_g, \mathbb{Z})$ as in the above corollary have an intersection pairing $\langle \cdot, \cdot \rangle : H_1(T_g, \mathbb{Z}) \times H_1(T_g, \mathbb{Z}) \to \mathbb{Z}$ given by

$$\begin{aligned} \langle \alpha_i, \alpha_{i+1} \rangle &= -1, \ 1 \leq i \leq 2g+1 \\ \langle \alpha_i, \alpha_j \rangle &= 0, \ 1 \leq i, j \leq 2g, \ |i-j| \geq 2. \end{aligned}$$

Let J be the $2g \times 2g$ matrix of this intersection pairing; i.e. for $1 \leq i, j \leq 2g$, $J_{i,j} = \langle \alpha_i, \alpha_j \rangle$.

It is easy to check in this case that the action of B_{2g+1} on $H_1(T_g, \mathbb{Z})$ as described in Corollary 2.2 preserves this intersection pairing. We may view it as a homomorphism taking B_{2g+1} to $\mathrm{Aut}(H_1(T_g, \mathbb{Z}))$. The choice of $\{\alpha_1, ..., \alpha_{2g}\}$ as an ordered basis of $H_1(T_g, \mathbb{Z})$ determines an isomorphism $\mathrm{Aut}(H_1(T_g, \mathbb{Z})) \cong \mathrm{GL}_{2g}(\mathbb{Z})$. Moreover, since the action of B_{2g+1} preserves the intersection pairing, its image in $\mathrm{GL}_{2g}(\mathbb{Z})$ actually lands in the symplectic group

$$\mathrm{Sp}_{2g}(\mathbb{Z}) := \{A \in \mathrm{GL}_{2g}(\mathbb{Z}) \mid A^T J A = J\}.$$

We denote this representation by $R_g : B_{2g+1} \to \mathrm{Sp}_{2g}(\mathbb{Z})$ and call it *the symplectic representation* of B_{2g+1}.

Remark 2.3. *For any element $x \in H_1(T_g, \mathbb{Z})$, we let $T_x \in \mathrm{Aut}(H_1(T_g, \mathbb{Z}))$ denote the transvection defined by $y \mapsto y + \langle y, x\rangle x$ for each $y \in H_1(T_g, \mathbb{Z})$. Then it is easy to check that $R_g(\beta_i) = T_{-\alpha_i}$ for $1 \leq i \leq 2g$.*

3 Dehn Twists

Let γ be a simple closed loop on a compact Riemann surface X. Think of it as (homeomorphic to) a copy of the unit circle S^1. Take a small tubular neighborhood around γ on X, which we think of as (homeomorphic to) $S^1 \times [-\epsilon, \epsilon]$. Each element of $S^1 \times [-\epsilon, \epsilon]$ can be written as (θ, y), where θ represents an angle, and $y \in [-\epsilon, \epsilon]$. The subset γ is $S^1 \times \{0\}$ is the original path γ, $S^1 \times \{-\epsilon\}$ is the "outer circle" and $S^1 \times \{\epsilon\}$ is the "inner circle". Let D_γ denote the homeomorphism from X to itself acting as the identity on $X \backslash S^1 \times [-\epsilon, \epsilon]$, and taking $(\theta, y) \in S^1 \times [-\epsilon, \epsilon]$ to $(\theta + (1 + \frac{s}{\epsilon})\pi, s)$. We may visualize D_γ as a self-homeomorphism that keeps the outer edge of the tubular neighborhood fixed while twisting the inner edge one full rotation counterclockwise. It is called a *Dehn twist* on X about the loop γ. If $X = T_g$ for some $g \geq 0$, then D_γ clearly is a representative of an element of the mapping class group $M_0(T_g) = M_0^0(T_g)$, and we will write $[D_\gamma] \in M_0^0(T_g)$. If $X = T_g$ is given n distinguished points none of which lie on γ, then clearly $[D_\gamma] \in M_n^0(T_g)$.

The following properties of Dehn twists are (easily) proven in [3].

Fact 3.1. *a) If γ is a simple closed loop on an oriented surface X, then D_γ does not depend on the orientation of γ.*

b) Let X be a compact Riemann surface with n distinguished points $z_1, ..., z_n$ for some $n \geq 0$. If γ_1 and γ_2 are isotopic simple closed loops on $X \backslash \{z_1, ..., z_n\}$, then $[D_{\gamma_1}] = [D_{\gamma_2}] \in M_n^0(X)$.

c) Let γ and c be simple closed loops on T_g. The Dehn twist D_γ acts on the homology class of c by $[c] \mapsto [c] + \langle [c], [\gamma]\rangle [\gamma]$. In other words, D_γ acts on $H_1(T_g, \mathbb{Z})$ as the transvection $T_{[\gamma]}$.

d) Dehn twists have infinite order.

We see from 3.1(c) that the all Dehn twists of a compact Riemann surface X act on the homology group $H_1(T_g, \mathbb{Z})$. Thus, there is an action of the subgroup of the mapping class group $M_0(T_g)$ generated by Dehn twists on $H_1(T_g, \mathbb{Z})$. But in fact, Dehn and Lickorish proved that the entire mapping class group of a compact Riemann surface is generated by Dehn twists (the proof can be found in [1]).

Theorem 3.2. *The mapping class group $M_0(T_g)$ can be generated by Dehn twists about a finite number of simple closed loops on T_g.*

A full set of relations between any two Dehn twists has also been determined ([3]).

Theorem 3.3. *Let γ_1 and γ_2 be two closed loops on T_g. Then,*

a) D_{γ_1} and D_{γ_2} commute if $\langle \gamma_1, \gamma_2 \rangle = 0$.

b) We have the one relation $D_{\gamma_1} D_{\gamma_2} D_{\gamma_1} = D_{\gamma_2} D_{\gamma_1} D_{\gamma_2}$ (the "braid relation") if $\langle \gamma_1, \gamma_2 \rangle = \pm 1$.

c) There are no relations between D_{γ_1} and D_{γ_2} if $|\langle \gamma_1, \gamma_2 \rangle| \geq 2$.

In this way, we obtain a representation $\tilde{R}_g : M_0(T_g) \to \mathrm{Aut}(H_1(T_g, \mathbb{Z}))$. It is proven in [2] that this representation is surjective. However, it is not faithful in general, and it would be interesting to identify the kernel of $\tilde{R}_g$, which is called *the Torelli subgroup*. Birman has determined a set of normal generators and relations for the Torelli subgroup for all $g \geq 1$ in [2]. These presentations are particularly simple for $g = 1$ and $g = 2$, and will be given in the examples below.

One particular application of the theory of Dehn twists is to find the image in $\mathrm{Sp}_{2g}(\mathbb{Z})$ of the pure braid group P_{2g+1} under R_g. The following theorem answering this question is in [4]. The proof given here is based on the proof in [4].

Theorem 3.4. *For any positive integer N, let $\Gamma_{2g}(N)$ denote the subgroup of matrices in $\mathrm{Sp}_{2g}(\mathbb{Z})$ which are congruent to I_{2g} modulo N. Then*

$$R_g(P_{2g+1}) = \Gamma_{2g}(2).$$

Proof. First we show that $R_g(B_{2g+1}) \leq \Gamma_{2g}(2)$. We see this by examining the homology groups of T_g, $T_g \backslash B$, and $T_0 \backslash (Z \cup \{\infty\}) = P \backslash Z$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$. First of all, since $T_g \backslash B$ is a cover of $P \backslash Z$, there is a map $H_1(T_g \backslash B, \mathbb{Z}) \to H_1(P \backslash Z, \mathbb{Z})$. As in Section 2, we will denote the generators of $\pi_1(P \backslash Z) = \pi_1(P \backslash Z)$ by $x_1, ..., x_{2g+1}$. Then, as in Section 2, $\pi_1(T_g \backslash B)$ is generated by x_i^2 for $1 \leq i \leq 2g+1$ and $x_i x_{i+1}$ for $1 \leq i \leq 2g$. Thus, the injection of fundamental groups (and hence, the injection of their abelianizations, the homology groups), is made explicit. It is clear that if we reduce to coefficients in $\mathbb{Z}/2\mathbb{Z}$, the induced map $H_1(T_g \backslash B, \mathbb{Z}/2\mathbb{Z}) \to H_1(P \backslash Z)$ has kernel generated by the $\bar{x}_i^2$'s (letting $\bar{x}_i$ be the image of x_i for each i). Thus,

$$H_1(T_g, \mathbb{Z}/2\mathbb{Z}) \cong H_1(T_g \backslash Z, \mathbb{Z}/2\mathbb{Z}) / \langle \bar{x}_1^2, ..., \bar{x}_{2g+1}^2 \rangle \hookrightarrow H_1(P \backslash Z, \mathbb{Z}/2\mathbb{Z})$$

is an injection. Now the action of the generators $\beta_1, ..., \beta_{2g}$ of B_{2g+1} on the generators $x_1, ..., x_{2g+1}$ of $\pi_1(P\backslash Z)$ was calculated in the proof of Proposition 2.1. It is clear that on the abelianization of $\pi_1(P\backslash Z)$, each β_i switches the images of x_i and x_{i+1} while leaving the images of the other generators fixed. In other words, each braid in B_{2g+1} acts on the images of $x_1, ..., x_{2g+1}$ via the permutation on $2g+1$ objects induced by the braid. Since P_{2g+1} is the subgroup of braids whose induced permutation is the identity, each braid in P_{2g+1} fixes the images of the x_i's and therefore acts as the identity on $H_1(P\backslash Z, \mathbb{Z}/2\mathbb{Z})$. The injection $H_1(T_g, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H_1(P\backslash Z, \mathbb{Z}/2\mathbb{Z})$ is equivarient with respect to this braiding action, so every braid in P_{2g+1} acts as the identity on $H_1(T_g, \mathbb{Z}/2\mathbb{Z})$ also. But this means that the image of every element of P_{2g+1} is a matrix in $\mathrm{Sp}_{2g}(\mathbb{Z})$ which is the identity when reduced modulo 2, so $R_g(P_{2g+1}) \leq \Gamma_{2g}(2)$.

Now we show that $\Gamma_{2g}(2) \leq R_g(B_{2g+1})$. In the statement of Corollary 2.2, the basis $\{\alpha_1, ..., \alpha_{2g}\}$ was specified; now we define a new basis $\{a_1, ..., a_g, b_1, ..., b_g\}$ by

$$a_i := \alpha_{2i-1},\ b_i := \alpha_{2i} + \alpha_{2i+2} + ... + \alpha_{2g}.$$

It is well known that $\Gamma_{2g}(2)$ can be generated by transvections, and that in fact, it may be generated by the set of $2g^2 + g$ generators

$$\{T^2_{2a_i}\},\ \{T^2_{2b_i}\},\ \{T_{2(a_i+a_j)}\}_{i<j},\ \{T_{2(b_i-b_j)}\}_{i<j},\ \{T_{2(a_i+b_j)}\}.$$

Thus, it suffices to show that each type of generator is the image under R_g of some element of P_{2g+1}.

Corollary 1.6 tells us that each homeomorphism in $M^0_{2g+2}(T_0)$ can be pulled back to an element of $P_{2g+2}(T_0)$ (which, in turn, can be pulled back to something in P_{2g+1}). In particular, a Dehn twist around a loop in $P\backslash Z$ (which is a homeomorphism of T_0 fixing ∞ and the points in Z) can be pulled back to an element of P_{2g+1}. Thus, it suffices to show that for each of the above generators, there is such a Dehn twist on $P\backslash Z$ which lifts to a self-homeomorphism of T_g that acts on the homology group of T_g as that generator.

For each generator T_{2a_i}, choose a loop in $P\backslash Z$ wrapping around z_{2i-1} and z_{2i} (but no other z_j). This loop lifts to two loops in T_g, each homotopic to $\pm a_i$. Thus, the Dehn twist about the chosen loop on $P\backslash Z$ lifts to the product of Dehn twists about a_i on T_g (remember from Fact 3.1(a) that the

orientation doesn't matter). By 3.1(c), the element $D_{a_i}^2$ acts on the homology group as $T_{a_i}^2 = T_{2a_i}$.

For each generator T_{2b_i}, we follow the same procedure with the loop in $P\backslash Z$ wrapping around the points z_{2i} through z_{2g+1}.

For each generator $T_{2(a_i+a_j)}$, we follow the same procedure with the loop in $P\backslash Z$ wrapping around z_{2i-1}, z_{2i}, z_{2j-1}, and z_{2j}.

For each generator $T_{2(b_i-b_j)}$, we follow the same procedure with the loop in $P\backslash Z$ wrapping around the points z_{2i} through z_{2j-1}.

For each generator $T_{2(a_i+b_j)}$, we follow the same procedure with the loop in $P\backslash Z$ wrapping around z_{2i-1} and then around the points z_{2i+1} through z_{2g+1}.

□

Example 3.5. ***g = 1***

We use the above theory to get an idea of what is going on in the elliptic curve case. In this case, we are examining the representation

$$R_1 : B_3 \to \mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_{2g}(\mathbb{Z}).$$

The braid group B_3 is generated by β_1 and β_2 with the single relation $\beta_1\beta_2\beta_1 = \beta_2\beta_1\beta_2$. Using Corollary 2.2, one computes that

$$R_1(\beta_1) = S_1 := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \; R_1(\beta_2) = S_2 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

It is well known that $\mathrm{SL}_2(\mathbb{Z})$ is generated by S_1 and S_2, so the image of B_3 under R_1 is all of $\mathrm{SL}_2(\mathbb{Z})$. The group $\mathrm{SL}_2(\mathbb{Z}) = \langle S_1, S_2 \rangle$ has the relations $S_1S_2S_1 = S_2S_1S_2$ and $(S_1S_2)^6 = 1$. The first relation lifts to the braid relation in S_3, but the second relation does not lift. Therefore, the kernel of R_1 is normally generated by $(\beta_1\beta_2)^6$. But since $(\beta_1\beta_2)^6 \in Z(B_3) = \langle (\beta_1\beta_2)^3 \rangle$ by 1.1(c), the kernel of R_1 is simply $\langle (\beta_1\beta_2)^6 \rangle$.

We may calculate the images of the generators of P_3 as follows:

$$R_1(A_{1,2}) = R_1(\beta_1)^2 = S_1^2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \; R_1(A_{2,3}) = R_1(\beta_2)^2 = S_2^2 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix},$$

$$R_1(A_{1,3}) = R_1(\beta_2)R_1(\beta_1)^2R_1(\beta_2)^{-1} = S_2S_1^2S_2^{-1} = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}.$$

Since we know from Theorem 3.4 that $R_1(B_3) = \Gamma_1(2)$, the above three matrices form a generating set for $\Gamma_1(2)$ (a set of relations could be obtained via the relations on the generators of P_3 along with the generator $(A_{1,2}A_{1,3}A_{2,3})^2$ of $\ker(R_1)$). But this is basically another way of looking at what is going on in the proof of Theorem 3.4. Namely, the braids $A_{1,2}$, $A_{1,3}$, and $A_{2,3}$ get mapped to $M_4^0(T_0)$ via $\delta_{0,4}$, and a representative for each $\delta_{0,4}(A_{i,j}) \in M_4^0(T_0)$ is the Dehn twist around a loop wrapping around z_i and z_j but not the other two points, for $1 \leq i < j \leq 3$. But letting $g = 1$ in the proof of Theorem 3.4, it is explained that for $(i, j) = (1, 2)$, the Dehn twist about this loop lifts to the square of the Dehn twist about $a_1 = \alpha_1$; for $(i, j) = (1, 3)$, the Dehn twist about this loop lifts to the square of the Dehn twist about $a_1 + b_1 = \alpha_1 + \alpha_2$, and for $(i, j) = (2, 3)$, the Dehn twist about this loop lifts to the square of the Dehn twist about $b_1 = \alpha_2$. Thus, they act on the homology by $T_{2\alpha_1}$, $T_{2(\alpha_1+\alpha_2)}$, and $T_{2\alpha_2}$ respectively. These are the matrices S_1^2, $S_2S_1^2S_2^{-1}$, and S_2^2 respectively.

Meanwhile, the mapping class group $M_0(T_1)$ is generated by a finite set of Dehn twists. By a stronger version of Theorem 3.2 provided in [1] applied to a genus-1 torus, a sufficient set of Dehn twists is D_{α_1} and D_{α_2}. By Theorem 3.3, the only relation between them is $D_{\alpha_1}D_{\alpha_2}D_{\alpha_1} = D_{\alpha_2}D_{\alpha_1}D_{\alpha_2}$. Note that D_{α_1} (respectively D_{α_2}) acts on homology as T_{α_1} (respectively T_{α_2}), which when realized as a matrix is S_1^{-1} (respectively S_2^{-1}). Since S_1 and S_2 have the additional relation $(S_1S_2)^6 = 1$, the Torelli subgroup of $M_0(T_1)$ is $\langle (D_{\alpha_1}^{-1}D_{\alpha_2}^{-1})^6 \rangle = \langle (D_{\alpha_1}D_{\alpha_2})^6 \rangle$ (infinite cyclic).

Example 3.6. ***g = 2***

In this case, presentations for the mapping class group $M_0(T_2)$, its Torelli subgroup, and the image $\mathrm{Sp}_4(\mathbb{Z})$ of its action on the homology group $H_1(T_2, \mathbb{Z})$ are also relatively simple. The (sometimes rather messy) proofs of the following results can be found in [1] and [2].

Dehn and Lickorish proved, in their stronger version of Theorem 3.2, that $M_0(T_2)$ can be generated by 5 Dehn twists, namely the Dehn twists about the loops α_1, α_2, α_3, α_4, and $\alpha_5 := -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$. Birman proved that a full set of relations is

$$[D_{\alpha_i}, D_{\alpha_j}] = 1,\ |i - j| \geq 2,\ 1 \leq i, j \leq 5,$$

$$D_{\alpha_i}D_{\alpha_{i+1}}D_{\alpha_i} = D_{\alpha_{i+1}}D_{\alpha_i}D_{\alpha_{i+1}},\ 1 \leq i \leq 4,$$

$$(D_{\alpha_1}...D_{\alpha_5})^6 = 1,$$

$$(D_{\alpha_1}...D_{\alpha_4}D_{\alpha_5}^2D_{\alpha_4}...D_{\alpha_1})^2 = 1,$$
$$[D_{\alpha_1}...D_{\alpha_4}D_{\alpha_5}^2D_{\alpha_4}...D_{\alpha_1}, D_{\alpha_i}] = 1,\ 1 \leq i \leq 5.$$

The Torelli subgroup is the normal closure of $\langle (D_{\alpha_1}D_{\alpha_2+\alpha_4})^6 \rangle$ (which is also the normal closure of $\langle (D_{\alpha_3}D_{\alpha_4})^6 \rangle$). Note that with respect to the basis $\{a_1, b_1, a_2, b_2\}$, the matrix of $D_{\alpha_1} = D_{a_1}$ is the block diagonal matrix with S_1 in the upper left and I_2 in the lower right, while the matrix of $D_{\alpha_2+\alpha_4} = D_{b_1}$ is the block diagonal matrix with S_2 in the upper left and I_2 in the lower right. It is obvious by looking at them this way that they satisfy the braid relation (same with D_{α_3} and D_{α_4}).

(Note: what is missing here is how to write $D_{\alpha_2+\alpha_4}$ in terms of the D_{α_i}'s, which were the generators given for the presentation of $M_0(T_2)$.)

References

[1] Birman, Joan. *Braids, Links, and Mapping Class Groups*, Annals of Mathematical Studies, no. 82 (1974).

[2] Birman, Joan. *On Siegel's Modular Group*, Annals of Mathematical Studies, no. 191 (1971).

[3] Farb, Benson; Margalit, Dan. *A Primer on Mapping Class Groups*, Princeton Mathematical Series, 49 (2012).

[4] Mumford, David. *Tata lectures on Theta, II*, Birkhauser (1982).