Siegel Moduli Space

Seminar for the course:

"Complex Abelian Varieties and Elliptic Curves"

Prof. Jeffrey Yelton
Cipriano Jr. Cioffo
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Contents

1 Introduction

In the following notes some classical results of Moduli Spaces are discussed. At first, the definitions of Moduli problems, fine and coarse moduli spaces and examples are given. Afterwards, the case of principally polarized abelian monifolds is considered. The Siegel moduli space is a coarse moduli space for such families but, a slight modification (adding a so called level(n)-structure) will give a fine moduli space for p.p.a.m..

2 Moduli Problems, fine and coarse moduli spaces and examples.

Intuitively, a moduli problem is a classification task. Families of considered objects are collected up to isomorphisms; a moduli space is an object which has an appropriate universal property respect to classes of families.

Definition 1. *Let C be a category, a* ***moduli problem*** *is a controvariant functor $F : C^{op} \to Set$.*

Classicals moduli problems are those where C is the category of *Schemes, complex manifolds, complex analytic spaces, etc..*

Definition 2. *A moduli problem is said representable if it exists an element* $\tilde{c} \in C$ *such that* F *is naturally isomorphic to the functor* $Hom_C(-, \tilde{c})$*;* $\tilde{c}$ *is called a* ***fine moduli space*** *for the moduli problem* F.

Definition 3. *A* ***coarse moduli space*** *is an element* $\bar{c} \in C$ *such that it exists a natural transformation* $F \Rightarrow Hom_C(-, \bar{c})$ *with the properties:*
i) for every other $d \in C$ *and natural transformation* $F \Rightarrow Hom_C(-, d)$*, it exists a unique arrow* $c \to d$ *which makes the obvious diagram commutative.*
ii) The map $F(\star) \to Hom_C(\star, \bar{c})$ *is an isomorphism, where* $\star$ *is the initial object of* C.

For those who are familiar with S. MacLane' book, a coarse moduli space is an initial arrow from F to the functor which takes an element $d \in C$ to the controvariant functor $Hom_C(-, d)$. Moreover, observe that a fine moduli space is unique up to isomorphisms and, a fine moduli space is obviously a coarse moduli space.

From now on, the moduli problems considered will be isomorphism classes of families of objects on objects of a given category. For instance, a *family* ξ *of elliptic curves on a complex manifold* T is a proper holomorphic map $p : \xi \to T$, such that each fiber $\xi_t := p^{-1}(t)$ is an elliptic curve, and it exists a section which gives the marked points of the elliptic curves.
Let $Ell : Mfld_{\mathbb{C}} \to Set$ the functor which takes a complex manifold T to the isomorphism class of families of elliptic curves on T, and it takes a morphism $f : S \to T$ to the map $Ell(T) \to Ell(S)$ which takes an equivalence class $\pi : \xi \to T$ to the equivalence class of the pullback family defined as follows:

$$f^*\xi = \{(s, x) \in S \times \xi : f(s) = \pi(x)\} \to S.$$

Remark 1. *It is an easy exercise to show that for moduli problems of families, a fine moduli space is equivalent to a manifold* M *and a family* $U \in Ell(M)$ *which has the following universal property: for any manifold* T *and* $\xi \in Ell(T)$*, there exists a unique map* $f : T \to M$ *such that the pullback of* U *is isomorphic to* ξ*, i.e.* $f^*U \cong \xi$*, such a family is called the universal family (it is unique up to isomorphism).*

Observation 1. *A fine moduli space is a rigid structure which often could not exist. Problems could arise when the objects of families under consideration have non-trivial automorphisms. In fact, if a curve* ξ_0 *has a non-trivial isomorphism, it could be possible to build a family* $\xi \to T$ *such that the fibers* $\xi_t \cong \xi_0$ *but* $\xi \ncong T \times \xi_0$*. Thus, if* $U \in Ell(M)$ *is a fine moduli space, it exists a unique map* $h : T \to M$ *such that* $\xi \cong h^*U = T \times \xi_0$*, which is absurde.*

Example 1. *Consider a curve* ξ_0 *and a non-trivial automorphism* σ*. Define an action of* $\mathbb{Z}$ *on* $\mathbb{C} \times \xi_0$*:*

$$n * (z, x) = (z + 2in\pi, \sigma^n x).$$

The quotient of $\mathbb{C}$ by the action of $\mathbb{Z}$ is equal to $\mathbb{C}^*$. Moreover, the family $(\mathbb{C} \times \xi_0)/\mathbb{Z} \to \mathbb{C}^*$ has fibers isomorphic to ξ_0 but it is not isomorphic to $\mathbb{C}^* \times \xi_0 \to \mathbb{C}^*$, otherwise the automorphism σ would be trivial, which is an absurde.

The above example and observation imply the following lemma:

Lemma 1. *Elliptic curves do not have a fine moduli space.*

Moreover, as it will be shown later elliptic curves have a coarse moduli space:

Lemma 2. $\mathbb{C}$ *is a coarse moduli space for elliptic curves.*

An useful strategy to trivialise automorphisms will be to add a so called level(n)-structure, as it will be shown in next sections.

3 Principally polarized abelian manifolds.

Recall that a *complex torus* T of dimension g is given by V/Λ, where V is a $\mathbb{C}$-vector space of dimension g and Λ is a full lattice of rank 2g.

Definition 4. *A hermitian form on V is a map $H : V \times V \to \mathbb{C}$ which is $\mathbb{C}$-linear in the first argument and $H(u,v) = \overline{H(v,u)}$, $\forall u, v \in V$. If H is an hermitian form, let define $E := \Im H$; then, H is called a Riemann form if $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$.*
Two Riemann form H_1, H_2 are said equivalent if there are $n_1, n_2 \in \mathbb{N}$, such that $n_1 H_1 = n_2 H_2$. $\tilde{H}$ will denote the equivalence class of H.

Note that E is $\mathbb{R}$-bilinear, antisymmetric and $E(z,w) = E(iz, iw)$.

Definition 5. *An abelian manifold is a complex torus which has a positive definite Riemann form.*

Due to the "main theorem of complex tori/ abelian varieties", abelian manifolds and abelian varieties are the same:

Theorem 1. *A complex torus $X \cong V/\Lambda$ is an abelian manifold if and only if it has the structure of an abelian variety.*

Definition 6. *A polarized abelian manifold is a pair $(X, \tilde{H})$, where X is an abelian manifold and $\tilde{H}$ is the equivalence class of the positive Riemann form of X. $\tilde{H}$ is called (homogeneous) polarization of X.*

A map between polarized abelian manifolds $\phi : (X_1.\tilde{H}_1) \to (X_2, \tilde{H}_2)$ is a holomorphic map (which is a homomorphism of groups) between the complex tori $\phi : X_1 \to X_2$ such that the lift $\tilde{\phi} : V_1 \to V_2$ is $\mathbb{C}$-linear and $\tilde{\phi}^\star H_2$ is equivalent to H_1.

The following theorem, which states a fundamental property of polarized abelian manifolds, will be useful in next section where it will be discussed fine and coarse moduli spaces of these objects; this result is important from the point of view of the observation in last section.

Theorem 2. *The automorphism group $Aut(X, \tilde{H})$ of every polarized abelian manifold $(X, \tilde{H})$ is finite.*

Definition 7. *A polarized abelian manifold $(X, \tilde{H})$ is called principally polarized if it exists an element $\bar{H}$ in the polarization class, such that Pf(E)=1. It means that E is equal to*

$$\begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$$

with respecto to a symplectic basis $\{\lambda_1, ..., \lambda_{2g}\}$ of Λ.

Obeserve that every abelian manifold of dimension 1 (i.e. elliptic curves) is principally polarized, because of it has always a positive definitie Riemann form such that Pf(E)=1. In fact, it is an easy proof to show that if $\{\lambda_1 . \lambda_2\}$ is a basis of Λ such that $Im(\lambda_1/\lambda_2) > 0$ then, the unique $\mathbb{R}$-bilinear form $E : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ such that $E(\lambda_1, \lambda_2) = 1$ is the imaginary part of a positive definite Riemann form H.

Moreover, the following theorem states that up to isogenies polarized abelian manifolds are principally polarized.

Theorem 3. *Every polarized abelian manifold is isogeneous to a principally plorized abelian manifold.*

Let now define the set of isomorphism classes $\mathcal{A}_g$ of principally polarized abelian manifolds.

Case g=1: Let $\mathcal{H} = \{\tau \in \mathbb{C} : Im\tau > 0\}$ be the *Poincaré upper half plane*, and consider $SL_2(\mathbb{Z})$ the two dimensional special linear subgroup with integer coefficients. $SL_2(\mathbb{Z})$ acts on $\mathcal{H}$ as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \tau = \frac{a\tau + b}{c\tau + d}.$$

Recall that an elliptic curve is a complex tori of dimension one, which can be written as $X = \mathbb{C}/\Lambda = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, which is in turn isomorphic to $X_\tau := \mathbb{C}/\tau\mathbb{Z} + \mathbb{Z}$, for some τ such that $Im\tau > 0$. Thus, see that two ellipctic curves X_{τ_1}, X_{τ_2} are isomorphic if and only if it exists an element A in $SL_2(\mathbb{Z})$ such that $A * \tau_1 = \tau_2$, the following bijection of sets is proved:

$$\mathcal{A}_1 \cong SL_2(\mathbb{Z})\backslash\mathcal{H}.$$

Moreover, observe that, by the j-invariant, $\mathcal{A}_1 \cong \mathbb{C}$.
Let now consider the *principal congruence subgroup* $\Gamma(n) = ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/n\mathbb{Z}))$ which is equal to,

$$\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : a, d \equiv 1 \ (mod\ n), b, c \equiv 0 \ (mod\ n)\ \}.$$

For every elliptic curve X_τ the points $\{1/n, \tau/n\}$ form a basis of the n-torsion subgroup X_τ, and it is possible to prove that this additional structure is preserved by $\Gamma(n)$, i.e.:

Lemma 3. *Two elliptic curves with a choice of a basis of the n-torsion points* $[X_\tau, 1/n + \Lambda_\tau, \tau/n + \Lambda_\tau]$, $[X_{\tau'}, 1/n + \Lambda_{\tau'}, \tau'/n + \Lambda_{\tau'}]$ *are isomorphic if and only if it exists an element* $M \in \Gamma(n)$ *such that* $M * \tau' = \tau$, *and* $\gamma 1/n = 1/n$, $\gamma\tau/n = \tau'/n$ *(mod n), where* $\gamma := c\tau' + d$. *See for a complete treatment [DS05].*

A choice of a basis of n-torsion points $X_\tau[n]$ is also called a *level(n)-structure.* Denoting $\tilde{\mathcal{A}_1}$ the set of isomorphism classes of these elliptic curves, the above lemma states the following bijection

$$\tilde{\mathcal{A}_1} \cong \Gamma(n)\backslash\mathcal{H}.$$

Case g>1: The above results can be generalized for $g > 1$.
Following [Lan12], let $X = V/\Lambda$ be a pricipally polarized abelian manifold, $\{v_1, \ldots, v_g\}$ a basis of $V = \mathbb{C}^g$ and $\{\omega_1, \ldots, \omega_{2g}\}$ a symplectic basis of Λ. Then, X can be represented as a triple

$$(\mathbb{C}^g, \Omega, J)$$

where Ω in the $g \times 2g$ matrix which columns are ω_i, and

$$J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}.$$

A general discussion leads to the following lemma, given a full lattice Λ of $\mathbb{C}^g$ and let Ω be the matrix which columns are a basis of Λ:

Lemma 4. *A triple* $(\mathbb{C}^g, \Omega, P)$, *where* P *is an alternatig matrix, defines an abelian manifold if and only if:*
(R1) $\Omega P^{-1}{}^t\Omega = 0$,
(R2) $2i(\overline{\Omega} P^{-1}{}^t\Omega)^{-1} > 0$.

In fact, considering the unique bilinear alternating form $E(\Omega x, \Omega y) = x^t P y$, property (R1) is equivalent to the fact that $E(i\Omega x, \Omega y)$ is a symmetric form and, moreover, it is possible to show that the associated Riemann form $H(v, w) = E(iv, w) + iE(v, w)$ is represented by the matrix in (R2). Considering Ω as $[\Omega_1, \Omega_2]$ with $\Omega_i \in Mat_g(\mathbb{C})$ gives a practical rewriting of the above conditions:
(R1') $\Omega_2 \Omega_1^t - \Omega_1 \Omega_2^t = 0$,
(R2') $2i(\Omega_2 \overline{\Omega}_1^t - \Omega_1 \overline{\Omega}_2^t) > 0$.

Now let $\mathcal{H}_g$ be the *Siegel upper half plane*:

$$\mathcal{H}_g := \{\tau \in Mat_g(\mathbb{C}) : \tau = \tau^t, Im(\tau) > 0\},$$

and consider also:

$$\mathcal{R} := \{\Omega = [\Omega_1, \Omega_2] : (R1'), (R2') \text{ } holds\}$$
$$\mathrm{Sp}_{2g}(\mathbb{R}) := \{M \in Mat_{2g}(\mathbb{R}) : MJM^t = J\}.$$

Lemma 5. *Let $\Omega = [\Omega_1, \Omega_2] \in \mathcal{R}$, then:*
(i) If $\rho \in GL_g(\mathbb{C})$, then $\rho\Omega := [\rho\Omega_1, \rho\Omega_2] \in \mathcal{R}$.
(ii) If $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R})$, then $\Omega M := [\Omega_1 A + \Omega_2 C, \Omega_1 B + \Omega_2 D] \in \mathcal{R}$.
(iii) $\Omega_1, \Omega_2 \in GL_g(\mathbb{C})$.
(iv) $\Omega_2^{-1}\Omega_1 \in \mathcal{H}_g$.

By the above lemma it is possible to prove the following:

Proposition 1. *Every principally polarized abelian manifold $(\mathbb{C}^g, [\Omega_1, \Omega_2], J)$ is isomorphic to one of the form $(\mathbb{C}^g, [\tau, Id_g], J)$, for a $\tau \in \mathcal{H}_g$*

Proof. See that $\Omega_2 \in GL_g(\mathbb{C})$, the multiplication by Ω_2 is a change of base and so an isomorpfinsm such that, $\Omega_2^{-1}[\Omega_1, \Omega_2] = [\Omega_2^{-1}\Omega_1, Id_g] \in \mathcal{R}$. Moreover, the change of base does not affect the description of E. □

Now observe that $\mathrm{Sp}_{2g}(\mathbb{R})$ acts on $\mathcal{H}_g$ in the following way:

$$M * \tau = \begin{bmatrix} A & B \\ C & D \end{bmatrix} * \tau := (A\tau + C)(B\tau + D)^{-1} \text{ for } M \in \mathcal{R}, \tau \in \mathcal{H}_g.$$

In fact, if $\Omega := [\tau, Id_g] \in \mathcal{R}$ then the above lemma implies that
$[\tau A + C, \tau B + D] = \Omega M \in \mathcal{R}$. By (iv), $(\tau B + D)^{-1}(\tau A + C) \in \mathcal{H}_g$, thus, $M * \tau \in \mathcal{H}_g$ because $\mathcal{H}_g$ and $\mathrm{Sp}_{2g}(\mathbb{R})$ are closed under transposition. The following theorem follows:

Theorem 4. *There is a bijection $\mathcal{A}_g \cong \mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathcal{H}_g$.*

Proof. A principally polarized abelian manifold $(\mathbb{C}^g, [\Omega_1, \Omega_2], J)$ corresponds to $\tau = \Omega_2^{-1}\Omega_1 \in \mathcal{H}_g$. If $(\mathbb{C}^g, [\Omega_1, \Omega_2], J) \cong (\mathbb{C}^g, [\Omega'_1, \Omega'_2], J)$ it exists a matrix $M \in GL_{2g}(\mathbb{Z})$ such tath $\Omega' = \Omega M^t$. With respect to the basis $\{\omega'_1, \ldots, \omega'_{2g}\}$, E is represented by MJM^t. Thus, the representation of E is unchanged if and only if $MJM^t = J$, i.e. if $M \in SL_{2g}(\mathbb{Z})$. □

I follows a rough discussion on the generalization of the previuos results adding a *level(n)-structure* for p.p.a.m. of dimension $g > 1$.

A level(n)-structure for $X : \mathbb{C}^g/\Lambda$ is a chioce of symplectic basis

$$H^1(X, \mathbb{Z}/n\mathbb{Z}) \cong Hom(\pi_1(X), \mathbb{Z}/n\mathbb{Z}) \cong Hom(\Lambda, \mathbb{Z}/n\mathbb{Z})$$

The folowing proposition will be useful later when the moduli space of such objects will be discussed, this result avoids problems discussed in observation 1.

Proposition 2. *If $n \geq 3$, if γ is an automorphism of a principally polarized abelian manifod (X, H) which acts trivially on the lattice (modn), then $\gamma = 1$.*

Let define

$$\Gamma(n) = ker(\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})),$$

the following generalization holds:

Theorem 5. *There is a bijection $\tilde{A}_g \cong \Gamma(n)\backslash\mathcal{H}_g$.*

4 Siegel moduli space and fine moduli space for p.p.a.m.

In this section moduli problems of p.p.a.m. are discussed, the quotient spaces defined in the have the wanted structure.

Lemma 6. *Every discrete subgroup of $\mathrm{Sp}_{2g}(\mathbb{R})$ acts properly and discontinuosly on $\mathcal{H}_g$. In particular $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\Gamma(n)$.*

The following theorem can be found in the appendix of [BL13]:

Theorem 6. *Let $\mathfrak{X}$ be a complex analytic space and let G be a group acting properly and discontinuously on $\mathfrak{X}$, then the quotient $\mathfrak{X}/G$ is a complex analytic space.*

Corollary 1. *Let $\mathfrak{X}$ be a complex manifold and let G be a group acting freely and properly and discontinuously on $\mathfrak{X}$, then the quotient $\mathfrak{X}/G$ is a complex manifold.*

Thus, $\mathcal{A}_g$ is a complex analytic space and observation 1, theorem 2 and theorem 4 make $\mathcal{A}_g$ be a coarse moduli space for the moduli problem which takes a complex analytic space to the set of equivalence classes of families of p.p.a.m. defined on it.

Moreover, it is possible to show that $\Gamma(n)$ acts freely on $\mathcal{H}_g$ and thus $\tilde{\mathcal{A}}_g$ is a complex manifold. As mentioned above, proposition 2 avoids problems with non-trivial automorphisms . $\tilde{\mathcal{A}}_g$ is actually a fine moduli space for isomorphism classes of families of p.p.a.m. defined of complex manifols. Moreover, the universal family is:

$$\mathcal{H}_g \times \mathbb{C}^g / \Gamma(n) \rtimes \mathbb{Z}^g \to \mathcal{H}_g / \Gamma(n) \cong \tilde{\mathcal{A}}_g.$$

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