PRIME-TO-p ÉTALE FUNDAMENTAL GROUPS OF PUNCTURED PROJECTIVE LINES OVER STRICTLY HENSELIAN FIELDS

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Abstract. Let K be the fraction field of a strictly Henselian DVR of characteristic $p \geq 0$ with algebraic closure $\bar{K}$, and let $\alpha_1, ..., \alpha_d \in \mathbb{P}^1_K(K)$. In this paper, we give explicit generators and relations for the prime-to-p étale fundamental group of $\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}$ that depend (solely) on their intersection behavior. This is done by a comparison theorem that relates this situation to a topological one. Namely, let $a_1, ..., a_d$ be distinct power series in $\mathbb{C}[[x]]$ with the same intersection behavior as the α_i's, converging on an open disk centered at 0, and choose a point $z_0 \neq 0$ lying in this open disk. We compare the natural action of $\mathrm{Gal}(\bar{K}/K)$ on the prime-to-p étale fundamental group of $\mathbb{P}_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}$ to the topological action of looping z_0 around the origin on the fundamental group of $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{a_1(z_0), ..., a_d(z_0)\}$. This latter action is, in turn, interpreted in terms of Dehn twists. A corollary of this result is that every prime-to-p G-Galois cover of $\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}$ satisfies that its field of moduli (as a G-Galois cover) has degree over K dividing the exponent of $G/Z(G)$.

1. Introduction

Let R be any strictly Henselian discrete valuation ring of characteristic 0 with residue characteristic $p \geq 0$ and uniformizer π. Let K be the fraction field of R, and fix an algebraic closure $\bar{K}$ of K. Let $\alpha_1, ..., \alpha_d$ be K-points of $\mathbb{P}^1_K$, and let P be a geometric point lying over a K-point of $\mathbb{P}^1_K$ different from $\alpha_1, ..., \alpha_d$. We have the fundamental short exact sequence

$$1 \to \pi_1^{\text{ét}}(\mathbb{P}_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, P) \to \pi_1^{\text{ét}}(\mathbb{P}^1_K \smallsetminus \{\alpha_1, ..., \alpha_d\}, P) \to G_K \to 1. \tag{1}$$

The K-point P corresponds to a section $\mathrm{Spec}(K) \to \mathbb{P}^1_K \smallsetminus \{\alpha_1, ..., \alpha_d\}$, which induces a splitting of the short exact sequence (1). This gives rise to an algebraic monodromy action of G_K on $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, P)$, which we denote by $\rho_{\mathrm{alg}} : G_K \to \mathrm{Aut}(\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, P))$. We write $\bar{\rho}_{\mathrm{alg}} : G_K \to \mathrm{Out}(\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, P))$ for the induced outer action.

For any group G, we write $\widehat{G}$ for its profinite completion. For any profinite group G, let $G^{(p')}$ denote the maximal prime-to-p quotient of G if $p > 0$, and let $G^{(p')} = G$ if $p = 0$. Since $G^{(p')}$ is a characteristic quotient of G, any action on G induces an action on $G^{(p')}$. Let $\rho_{\mathrm{alg}}^{(p')} : G_K \to \mathrm{Aut}(\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, P))^{(p')})$ denote the action induced by ρ_{alg}, and let $\bar{\rho}_{\mathrm{alg}}^{(p')}$ be the induced outer action. While a complete description of $\bar{\rho}_{\mathrm{alg}}$ remains out of reach, the main theorem of this paper (Theorem 1.2), together with Theorem 2.3, gives a complete and explicit description of $\bar{\rho}_{\mathrm{alg}}^{(p')}$.

1.1. Comparison of topological and algebraic monodromy actions.

Fix an integer $d \geq 2$, and choose distinct power series $a_1, ..., a_d \in \mathbb{C}[[x]]$ with positive radii of convergence. For each i and any complex number $z \in \mathbb{C}$, we write for $a_i(z)$ for the evaluation of a_i at z; if a_i diverges at z, then we consider $a_i(z)$ to be the infinity point of $\mathbb{P}^1_{\mathbb{C}}$. For any real number $\varepsilon > 0$, we write $B_\varepsilon := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ for the (open) ball of radius ε and let $B_\varepsilon^* = B_\varepsilon \smallsetminus \{0\}$. We will always make the following assumption.

Hypothesis 1.1. The real number $\varepsilon > 0$ is small enough that each power series a_i converges on the closure of B_ε, and for any $z \in B_\varepsilon^*$, we have $a_i(z) \neq a_j(z)$ for $i \neq j$.

We define a family $\mathcal{F} \to B_\varepsilon^*$ of punctured Riemann spheres by letting

$$\mathcal{F} = ((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \times B_\varepsilon^*) \smallsetminus \bigcup_{i=1}^{d} \{(a_i(z), z) \mid z \in B_\varepsilon^*\}.$$

It follows from Hypothesis 1.1 that the obvious projection map $\mathcal{F} \to B_\varepsilon^*$ is a fiber bundle. For each $z \in B_\varepsilon^*$, write ∞_z for the point in the fiber $\mathcal{F}_z := (\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{a_1(z), ..., a_d(z)\}$. Since the punctured disk B_ε^* has trivial j^{th} homotopy group for $j \geq 2$, the associated long exact homotopy sequence of this fiber bundle truncates. Therefore, after choosing a basepoint $z_0 \in B_\varepsilon^*$, we have the short exact sequence of fundamental groups

$$1 \to \pi_1(\mathcal{F}_{z_0}, \infty_{z_0}) \to \pi_1(\mathcal{F}, \infty_{z_0}) \to \pi_1(B_\varepsilon^*, z_0) \to 1. \tag{2}$$

There is an obvious "infinity section" of the family $\mathcal{F} \to B_\varepsilon^*$ given by $z \mapsto \infty_z$. This section induces a splitting of the short exact sequence (2). This gives rise to a monodromy action of $\pi_1(B_\varepsilon^*, z_0)$ on $\pi_1(\mathcal{F}_{z_0}, \infty_{z_0})$, which we denote by $\rho_{\mathrm{top}} : \pi_1(B_\varepsilon^*, z_0) \to \mathrm{Aut}(\pi_1(\mathcal{F}_{z_0}, \infty_{z_0}))$.

We remark that taking profinite completions of the terms in the sequence (1) produces a short exact sequence. Indeed, the profinite completion functor is right exact ([RZ10, Proposition 3.2.5]); left-exactness of the resulting sequence now follows from [And74, Proposition 4] after observing that the groups appearing in (1) are finitely generated, the sequence is split, and $\pi_1(B_\varepsilon^*, z_0)$ is residually finite. We further remark that the natural maps from $\pi_1(B_\varepsilon^*, z_0)$ and $\pi_1(\mathcal{F}_{z_0}, \infty_{z_0})$ to their respective profinite completions are embeddings, as these are free groups and therefore residually finite. By the Five Lemma, it follows that the natural map from $\pi_1(\mathcal{F}, \infty_{z_0})$ to its profinite completion is also injective. It follows that ρ_{top} extends uniquely to a continuous action of $\widehat{\pi}_1(B_\varepsilon^*, z_0)$ on $\widehat{\pi}_1(\mathcal{F}_{z_0}, \infty_{z_0})$; we denote this action also by ρ_{top}. We write $\bar{\rho}_{\mathrm{top}} : \widehat{\pi}_1(B_\varepsilon^*, z_0) \to \mathrm{Out}(\widehat{\pi}_1(\mathcal{F}_{z_0}, \infty_{z_0}))$ for the induced outer monodromy action, and we write $\rho_{\mathrm{top}}^{(p')}$ (resp. $\bar{\rho}_{\mathrm{top}}^{(p')}$) for the action (resp. the outer action) induced by ρ_{top} on $\widehat{\pi}_1(\mathcal{F}_{z_0}, \infty_{z_0})^{(p')}$. For $1 \leq i < j \leq d$, we write $e_{i,j} = v_x(a_i - a_j)$, where v_x is the x-adic valuation, which is the intersection multiplicity of the power series a_i and a_j viewed as divisors on the surface $\mathbb{P}^1_{\mathbb{C}[[x]]}$. Meanwhile, since $\mathbb{P}^1_R$ is proper over $\mathrm{Spec}(R)$, the K-points $\alpha_i \in \mathbb{P}^1_K$ extend to R-points of $\mathbb{P}^1_R$, which we also denote by α_i. For $1 \leq i < j \leq d$, we write $E_{i,j}$ for the intersection multiplicity of the R-points α_i and α_j as divisors on the surface $\mathbb{P}^1_R$.

We may visualize $\mathrm{Spec}(R)$ as an infinitesimally small disk and $\mathrm{Spec}(K)$ as an infinitesimally small punctured disk. Our main theorem asserts that the action of G_K on $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, P)^{(p')}$ agrees with this intuition: namely, it is "the same" as the monodromy action of $\pi_1(B_\varepsilon^*, z_0)$ on $\pi_1(\mathcal{F}_{z_0}, \infty_{z_0})^{(p')}$ as long as the intersection behavior of the α_i's over $\mathrm{Spec}(R)$ is the same as the intersection behavior of the a_i's over B_ε.

Theorem 1.2. *In the above situation, we have the following.*

a) The actions $\rho_{\mathrm{top}}^{(p')}$ and $\rho_{\mathrm{alg}}^{(p')}$ factor through $\pi_1(B_\varepsilon^*, z_0)^{(p')}$ and $G_K^{(p')}$ respectively.

b) If $e_{i,j} = E_{i,j}$ for $1 \leq i < j \leq d$, then there exist isomorphisms $\widehat{\pi}_1(B_\varepsilon^*, z_0)^{(p')} \xrightarrow{\sim} G_K^{(p')}$ and $\widehat{\pi}_1(\mathcal{F}_{z_0}, \infty_{z_0})^{(p')} \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, P)^{(p')}$ inducing an isomorphism of the outer actions $\bar{\rho}_{\mathrm{top}}^{(p')}$ and $\bar{\rho}_{\mathrm{alg}}^{(p')}$. Moreover, this lifts to an isomorphism of the actions $\rho_{\mathrm{top}}^{(p')}$ and $\rho_{\mathrm{alg}}^{(p')}$ provided that $\alpha_i \in R \cup \{\infty\}$ for $1 \leq i \leq d$.

Remark 1.3. In [Yel17], the second author derives a particular application of the above theorem towards understanding the ℓ-adic Galois action action associated to the Jacobians of hyperelliptic curves over number fields, through viewing such a curve as a cover of the projective line ramified at certain points and localizing at primes $p \neq \ell, 2$ at which the curve has bad reduction.

Remark 1.4. Much of this paper was inspired by ideas in [Flo04], which gave a treatment, up to some imprecise language, of a problem that is very similar in spirit. In this paper, the authors have decided to follow a different approach, similar to that of Oda in [Oda95], which the authors believe to be more direct and more instructive to the reader. Namely, we reduce to the equicharacteristic case via a 2-dimensional ring (see §3.3) and then relate the topological action to the $R = \mathbb{C}[[x]]$ case directly.

1.2. **Outline of the paper.** In §2, we will use purely topological constructions and arguments to explicitly describe the action ρ_{top} (Theorem 2.3). The whole of §3 is dedicated to a proof of Theorem 1.2. In §4, we will construct a specific basis of $\pi_1(\mathcal{F}_{z_0}, \infty_{z_0})$ for which we can describe the action of ρ_{top} explicitly. Then we give some basic examples of how one can combine Theorem 1.2 with the explicit formulas given in §4.1 to give explicit generators and relations of the prime-to-p étale fundamental groups of $\mathbb{P}^1_K$ minus K-rational branch points. Our formulas will also be used to prove (Corollary 4.8) that the degree over K of the field of moduli, together with its group action, of any prime-to-p Galois cover of $\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}$, divides the exponent of G modulo its center.

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2. The topological monodromy action

We begin this section with a brief overview of some topological preliminaries which we will need below. We will then construct some simple loops whose images lie in $(\mathbb{P}^1_{\mathbb{C}})^{\text{an}} \smallsetminus \{a_1(z_0), ..., a_d(z_0)\}$ and present and prove Theorem 2.3, which describes the monodromy action ρ_{top} in terms of Dehn twists associated to these loops.

2.1. **Configuration spaces, mapping class groups, and Dehn twists.** For any integer $d \geq 2$, we define Y_d to be the complex manifold $\mathbb{C}^d \smallsetminus \bigcup_{1 \leq i < j \leq d} \Delta_{i,j}$, where each $\Delta_{i,j}$ is the "weak diagonal" subspace of $\mathbb{C}^d$ consisting of points $(z_1, ..., z_d)$ with $z_i = z_j$. We endow $Y_d \subset \mathbb{C}^d$ with the subspace topology and call it the *ordered configuration space* of d-element subsets of $\mathbb{C}$. Each point $(z_1, ..., z_d) \in Y_d$ may be identified with the ordered d-element subset $\{z_1, ..., z_d\} \subset \mathbb{C}$.

Given an integer $d \geq 2$ and any basepoint $T_0 \in Y_d$, we write $\mathcal{Y}_d$ for the group of self-homeomorphisms of the complex plane which fixes the subset $T_0 \subset \mathbb{C}$ pointwise (note that the structure of $\mathcal{Y}_d$ as an abstract topological group does not depend on our choice of T_0). Additionally, let $\mathcal{Y}_0$ denote the group of all self-homeomorphisms of the complex plane. We endow these groups of homeomorphisms with the compact-open topology and consider them as topological groups. Note that we have an obvious inclusion $\iota : \mathcal{Y}_d \hookrightarrow \mathcal{Y}_0$.

For $d \geq 2$, the *pure mapping class group (of the plane)* $\pi_0\mathcal{Y}_d$ is the quotient of the topological group $\mathcal{Y}_d$ modulo the path component of the identity $\text{id} \in \mathcal{Y}_d$. Note that each self-homeomorphism $f \in \mathcal{Y}_d$ induces an automorphism of $\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\text{an}} \smallsetminus T_0, \infty)$. Since any self-homeomorphism in $\mathcal{Y}_d$ can be extended uniquely to a self-homeomorphism of $(\mathbb{P}^1_{\mathbb{C}})^{\text{an}}$ which fixes the elements of T_0 as well as ∞, there is an obvious action of $\pi_0\mathcal{Y}_d$ on $\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\text{an}} \smallsetminus T_0, \infty)$, which we denote by $\varphi : \pi_0\mathcal{Y}_d \to \text{Aut}(\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\text{an}} \smallsetminus T_0, \infty))$.

Let $\gamma : [0, 1] \to (\mathbb{A}^1_{\mathbb{C}})^{\text{an}} \smallsetminus T_0$ be a simple loop, and write $\text{im}(\gamma)$ for its image in $(\mathbb{A}^1_{\mathbb{C}})^{\text{an}} \smallsetminus T_0$. The simple loop $\text{im}(\gamma)$ is homeomorphic to the unit circle $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, and this homeomorphism can be chosen such that the image of $\gamma(t)$ is $e^{2\pi\sqrt{-1}t} \in S^1$ for $t \in [0, 1]$. Take a small tubular neighborhood around $\text{im}(\gamma)$ on $(\mathbb{A}^1_{\mathbb{C}})^{\text{an}} \smallsetminus T_0$, which is homeomorphic to $S^1 \times [-\xi, \xi]$ for some small $\xi > 0$. By abuse of notation, we identify this neighborhood with $S^1 \times [-\xi, \xi]$; the outer edge is $S^1 \times \{\xi\}$, and the inner edge is $S^1 \times \{-\xi\}$. We define the *Dehn twist* $D_\gamma : (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \to (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$ to be the self-homeomorphism which acts as the identity on $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus S^1 \times [-\xi, \xi]$ and which takes $(e^{2\pi\sqrt{-1}t}, s) \in S^1 \times [-\xi, \xi]$ to $(e^{2\pi\sqrt{-1}(t+1/2-s/(2\xi))}, s)$. We may visualize D_γ as a self-homeomorphism of $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$ that keeps the outer edge of the tubular neighborhood fixed while twisting the inner edge one full rotation counterclockwise. Clearly, $D_\gamma \in \mathcal{Y}_d$; we also denote its path component in $\pi_0\mathcal{Y}_d$ by D_γ.

Let $\epsilon : \mathcal{Y}_0 \to Y_d$ be the *evaluation map* defined by sending any homeomorphism $f \in \mathcal{Y}_0$ to $(f(z_1), ..., f(z_d)) \in Y_d$. Note that the inverse image of $T_0 = (z_1, ..., z_d) \in Y_d$ under ϵ is $\mathcal{Y}_d$. It follows easily from [Bir74, Theorem 4.1] that this map is a fiber bundle. Therefore, ϵ induces a long exact sequence of fundamental groups

$$... \stackrel{\iota_*}{\to} \pi_1(\mathcal{Y}_0, \mathrm{id}) \stackrel{\epsilon_*}{\to} \pi_1(Y_d, T_0) \stackrel{\partial}{\to} \pi_0\mathcal{Y}_d \stackrel{\iota_*}{\to} \pi_0\mathcal{Y}_0 \stackrel{\epsilon_*}{\to} \pi_0 Y_d = 1. \tag{3}$$

We observe that by [Bir74, Theorem 4.4], the mapping class group $\pi_0\mathcal{Y}_0$ is trivial, and therefore the map $\partial : \pi_1(Y_d, T_0) \to \pi_0\mathcal{Y}_d$ is surjective. This map can be described explicitly as follows. Let $\gamma : [0, 1] \to Y_d$ be a loop, with $\gamma(0) = \gamma(1) = T_0$, and let $[\gamma] \in \pi_1(Y_d, T_0)$ be the corresponding equivalence class. Then γ lifts to a path $\tilde{\gamma} : [0, 1] \to \mathcal{Y}_0$ with $\tilde{\gamma}(0) = \mathrm{id}$, via the fiber bundle $\epsilon : \mathcal{Y}_0 \to Y_d$. Note that the self-homeomorphism $\tilde{\gamma}(1) : (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \to (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$ fixes the points $z_1, ..., z_d$ since $\gamma(1) = (z_1, ..., z_d)$, so $\tilde{\gamma}(1) \in \mathcal{Y}_d$. Then $\partial([\gamma]) \in \pi_0\mathcal{Y}_d$ is the path component of $\tilde{\gamma}(1)$.

Note that we have a surjective map $Y_{d+1} \to Y_d$ defined by "forgetting" the $(d+1)^{\mathrm{st}}$ point, under which the inverse image of any point $T \in Y_d$ is $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus T$. According to [FN62, Theorem 3], this map is a fiber bundle. It is also shown in [FN62] that Y_d (and Y_{d+1}) have trivial j^{th} homotopy groups for $j \geq 2$.

Now define the family $\bar{Y}_{d+1} \to Y_d$ to be the family $Y_{d+1} \to Y_d$ "with the infinity section added"; that is, $\bar{Y}_{d+1} \to Y_d$ is the subfamily of the isotrivial family $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \times Y_d \to Y_d$ such that over each $T \in Y_d$, the fiber $(\bar{Y}_{d+1})_T$ is $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus T$. For each $T \in Y_d$, write ∞_T for the point in $\bar{Y}_{d+1}$ coming from the point at infinity in the fiber $(\bar{Y}_{d+1})_T = (\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus T$. It is clear that $\bar{Y}_{d+1} \to Y_d$ is also a fiber bundle. Then the associated long exact homotopy sequence of the fiber bundle $\bar{Y}_{d+1} \to Y_d$ truncates, and so we have the short exact sequence of fundamental groups

$$1 \to \pi_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus T_0, \infty) \to \pi_1(\bar{Y}_{d+1}, \infty_{T_0}) \to \pi_1(Y_d, T_0) \to 1. \tag{4}$$

There is an obvious "infinity section" of the family $\bar{Y}_{d+1} \to Y_d$ given by $T \mapsto \infty_T$. This section induces a splitting of the short exact sequence (4). This gives rise to a monodromy action of $\pi_1(Y_d, T_0)$ on $\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus T_0, \infty)$, which we denote by $\rho : \pi_1(Y_d, T_0) \to \mathrm{Aut}(\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus T_0, \infty))$.

2.2. **Description of the topological monodromy action.** In Theorem 2.3 we will describe the topological monodromy action in terms of loops on $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{a_1(z_0), ..., a_d(z_0)\}$ that depend on the intersection behavior of the a_i's. These loops can be taken to be circles for a suitable choice of z_0. Since knowledge of the monodromy action does not depend on our choice of z_0, we will allow ourselves to restrict our attention to such a strategic choice.

We begin by introducing some notation. Let $F_n : \mathbb{C}[[x]] \to \mathbb{C}[x]$ be the map given by $\sum_{i=0}^{\infty} c_i x^i \mapsto \sum_{i=0}^{n-1} c_i x^i$. Let $\mathcal{I}$ be the set of all pairs (I, n) where $I \subseteq \{1, ..., d\}$ is a subset and $n \geq 1$ is an integer such that $e_{i,j} \geq n$ for all $i, j \in I$ and such that I is maximal among subsets with this property. We remark that if (I, n) is in $\mathcal{I}$, then for all $i, j \in I$ we have that $F_n(a_i) = F_n(a_j)$. For such a pair (I, n), we denote this polynomial by $b_{I,n}$, and let $w_{I,n} = b_{I,n}(z_0)$.

Let $\eta, r > 0$, and let $\gamma_{I,n}$ be the loop given by $t \mapsto w_{I,n} + r^{n-1}\eta e^{2\pi\sqrt{-1}t}$. We write $\mathrm{im}(\gamma_{I,n})$ for its image in $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$, and write $B_{I,n} := \{z \in (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \mid |z - w_{I,n}| < r^{n-1}\eta\}$ for the simply connected component of $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \mathrm{im}(\gamma_{I,n})$. Note that $\gamma_{I,n}$ depends also on η and r, but we suppress this in the notation.

Proposition 2.1. *Assume all of the above notation. For every $\eta > 0$ sufficiently small, there exists an $r > 0$ sufficiently small, so that if $\frac{r}{2} < |z_0| < r$ then the following hold:*

a) For distinct pairs $(I, n), (I', n') \in \mathcal{I}$, the images $\mathrm{im}(\gamma_{I,n})$ *and* $\mathrm{im}(\gamma_{I',n'})$ *do not intersect; and*
b) For any pair $(I, n) \in \mathcal{I}$, we have $a_i(z_0) \in B_{I,n}$ for $i \in I$ and $a_j(z_0) \notin B_{I,n} \cup \mathrm{im}(\gamma_{I,n})$ for $j \notin I$.

In order to prove this proposition, we require the following lemma.

Lemma 2.2. *For every $\eta > 0$, there exists an $r > 0$ sufficiently small such that for every $(I, n) \in \mathcal{I}$, and for every $i \in I$ and $z \in B_r$ we have*

$$|a_i(z) - b_{I,n}(z)| < |z|^{n-1}\eta.$$

Proof. Since $\mathcal{I}$ is finite, it suffices to prove the statement for a particular (I, n). Let $a_{i,n} = x^{-n}(a_i - b_{I,n})$, and note that it converges wherever a_i does. The continuity of the map $\mathbb{C} \to \mathbb{C}^{\#I}$ that takes $z \mapsto (z \cdot a_{i,n}(z))_{i\in I}$ implies that given $\eta > 0$, there is a $r > 0$ such that $|z \cdot a_{i,n}(z)| < \eta$ for $i \in I$ and any $z \in B_r$. Thus, we have $|a_i(z) - b_{I,n}(z)| = |z^n a_{i,n}(z)| = |z|^{n-1}|z \cdot a_{i,n}(z)| < |z|^{n-1}\eta$ for $i \in I$. □

We are now ready to prove Proposition 2.1.

Proof. (of Proposition 2.1)
Choose distinct pairs $(I, n), (I', n') \in \mathcal{I}$, and assume without loss of generality that I is not strictly contained in I'. It then follows from obvious properties of the intersection index that either $I \supseteq I'$ or $I \cap I' = \varnothing$.

First assume that $I \supseteq I'$. Then we may assume without loss of generality that $n < n'$ (which is automatically the case if $I \supsetneq I'$). Let z satisfy $|z - w_{I',n'}| = \eta r^{n'-1}$. Then we have

$$|z - w_{I,n}| \le |z - w_{I',n'}| + |w_{I,n} - w_{I',n'}| = \eta r^{n'-1} + |z_0|^n |z_0^{-n}(w_{I,n} - w_{I',n'})|$$

$$< \eta r^{n'-1} + r^n |z_0^{-n}(w_{I,n} - w_{I',n'})| \le (\eta + |z_0^{-n}(w_{I,n} - w_{I',n'})|)r^n, \tag{5}$$

where the last inequality holds if r is chosen to be less than 1. Since x^n divides $b_{I,n} - b_{I',n'}$, it follows that we may choose a sufficiently small r, depending on η, but independent of the choice of z_0, so that $(\eta + |z_0^{-n}(w_{I,n} - w_{I',n'})|)r < \eta$. Thus $|z - w_{I,n}| < \eta r^{n-1}$.

Now assume that $I \cap I' = \varnothing$. Given what we have proven above, we may assume without loss of generality that $n = n'$ and that x^{n-1} divides $b_{I,n} - b_{I',n}$. Since $b_{I,n}$ and $b_{I',n}$ both have degree at most $n - 1$, this implies that $b_{I,n} - b_{I',n} = cx^{n-1}$ for some nonzero constant c. For any choice of $\eta < 2^{-n}c$, we therefore have $|w_{I,n} - w_{I',n}| = |c||z_0|^{n-1} > \eta |z_0|^{n-1} 2^n$. Since we have assumed that $\frac{r}{2} < |z_0|$, it follows that $|w_{I,n} - w_{I',n}| > \eta(\frac{r}{2})^{n-1} 2^n = 2r^{n-1}\eta$. This proves part (a).

It follows immediately from Lemma 2.2 that we may choose r small enough so that for every $(I, n) \in \mathcal{I}$ and $i \in I$, the point $a_i(z_0)$ will be in $B_{I,n}$. Now choose $j \notin I$; to prove part (b), it suffices to show that $a_j(z_0) \notin B_{I,n} \cup \mathrm{im}(\gamma_{I,n})$. The maximality of I implies that $F_n(a_j) \neq b_{I,n}$, and therefore, x_n does not divide $a_j - b_{I,n}$. It follows that there is a constant $c' > 0$ such that $|a_j(z) - b_{I,n}(z)| \ge c'|z|^{n-1}$ for $z \in B_r^*$ if r is chosen to be small enough, independently of η. Assume that we have chosen η so that $\eta < 2^{-(n-1)}c'$. Then we have

$$|a_j(z_0) - w_{I,n}| = |a_j(z_0) - b_{I,n}(z_0)| \ge |z_0|^{n-1}c' \ge 2^{-(n-1)} r^{n-1} c' > \eta r^{n-1}. \tag{6}$$

It follows that $a_j(z_0) \notin B_{I,n} \cup \mathrm{im}(\gamma_{I,n})$, and part (b) is proved. □

Each $\gamma_{I,n}$ defined above induces a induces a Dehn twist which we denote by $D_{I,n} \in \pi_0\mathcal{Y}_n$. The proposition above implies that for every $\eta > 0$ small enough there exists an $r > 0$ small enough for which the the $D_{I,n}$'s commute.

For ease of notation, given any $t \in \mathbb{R}$, we write $e(t)$ to mean $e^{2\pi\sqrt{-1}t}$. Let $\delta \in \pi_1(B_\varepsilon^*, z_0)$ be the homotopy equivalence class of the loop given by $t \mapsto e(t)z_0$ for $t \in [0, 1]$. Clearly, δ is the generator of $\pi_1(B_\varepsilon^*, z_0) \cong \mathbb{Z}$. Therefore, in order to determine the monodromy action ρ_{top}, it suffices to know how δ acts on $\pi_1(\mathcal{F}_{z_0}, \infty)$.

We are finally ready to state our main topological result.

Theorem 2.3. *For every $\eta > 0$ sufficiently small, there exists an $r > 0$ sufficiently small so that if $\frac{r}{2} < |z_0| < r$, then the generator $\delta \in \pi_1(B_\varepsilon^*, z_0)$ acts via ρ_{top} on $\pi_1(\mathcal{F}_{z_0}, \infty_{z_0})$ in the same way that the product of Dehn twists $\prod_{(I,d)\in\mathcal{I}} D_{I,n}$ does; in other words, $\rho_{\mathrm{top}}(\delta) = \varphi(\prod_{(I,d)\in\mathcal{I}} D_{I,n})$.*

Remark 2.4.

(a) The above theorem implies in particular that in this situation, the monodromy action depends only on the intersection behavior of the power series over the disk B_ε.

(b) The above theorem can be viewed as a generalization of a result of Oda given by [Oda95, Main Lemma 1.7]. Oda's lemma states that under certain technical hypotheses regarding their degenerations, a generator of the monodromy action associated to a family of *compact* genus-g Riemann surfaces over the punctured disk acts as a certain product of Dehn twists. Indeed, for every $g \geq 0$ and compact genus-g Riemann surface there exists a Zariski open subset that can be viewed as degree-2 covering space of the sphere minus $2g + 2$ points.

2.3. **Proof of Theorem 2.3.** In order to prove Theorem 2.3, we first need several lemmata. To simplify notation in the lemma below, we make the following extra assumption regarding the power series a_i, which always holds after a suitable reordering.

Hypothesis 2.5. For every $m \in \{1, ..., d\}$, the function $\{m+1, ..., d\} \to \mathbb{Z}_{\geq 0}$ given by $i \mapsto e_{m,i}$ is (weakly) monotonically decreasing.

Note that this assumption implies that the subsets $I \subseteq \{1, ..., d\}$ such that $(I, n) \in \mathcal{I}$ for some $n \geq 1$ are always subintervals $\{m, ..., m+l-1\}$ with $1 \leq m \leq d$ and $l \geq 2$.

For $I = \{m, ..., m+l-1\}$ with $(I, n) \in \mathcal{I}$, let $\lambda_{I,n}$ be the loop in Y_d given by:

$$(((a_i(z_0))_{1\leq i\leq m-1}, (w_{I,n} + e(t)(a_i(z_0) - w_{I,n}))_{m\leq i\leq m+l-1}, ((a_i(z_0))_{m+l\leq i\leq d}))_{0\leq t\leq 1})$$

Lemma 2.6. *Let $J = \{m, ..., m+l-1\}$ be a subinterval such that $(J, 1) \in \mathcal{I}$. For each $i \in J$, let $c_i(x) = x^{-1}(a_i(x) - w) \in \mathbb{C}[[x]]$, and let $w = a_m(0) = ... = a_{m+l-1}(0)$. Let $r' > 0$ be small enough so that for all $z \in B_{r'}$ and $i = 1, ..., d$ we have $|a_i(z) - a_i(0)| < \mu := \frac{1}{2}\min_{i\notin J} |a_i(0) - w|$. Then the loop in Y_d given by*

$$(a_1(e(t)z_0), ..., a_d(e(t)z_0))_{0\leq t\leq 1}$$

is homotopic to the concatenation of $\lambda_{J,1}$ with the loop in Y_d given by

$$((a_i(e(t)z_0))_{1\leq i\leq m-1}, (w + z_0c_i(e(t)z_0))_{m\leq i\leq m+l-1}, (a_i(e(t)z_0))_{m+l\leq i\leq d})_{0\leq t\leq 1}.$$

Proof. We define the continuous map $H : [0,1] \times [0,1] \to \mathbb{C}^d$ as follows.

$$H(s,t) = \begin{cases} \Big((a_i(z_0))_{1\leq i\leq m-1}, \\ \quad (w + e(\frac{t}{1-s/2})z_0c_i(z_0))_{m\leq i\leq m+l-1}, ((a_i(z_0))_{m+l\leq i\leq d}\Big) & 0 \leq t \leq s/2 \\ \Big((a_i(e(\frac{t-s/2}{1-s/2})z_0))_{1\leq i\leq m-1}, \\ \quad (w + e(\frac{t}{1-s/2})z_0c_i(e(\frac{t-s/2}{1-s/2})z_0))_{m\leq i\leq m+l-1}, (a_i(e(\frac{t-s/2}{1-s/2})z_0))_{m+l\leq i\leq d}\Big) & s/2 \leq t \leq 1 - s/2 \\ \Big((a_i(e(\frac{t-s/2}{1-s/2})z_0))_{1\leq i\leq m-1}, \\ \quad (w + z_0c_i(e(\frac{t-s/2}{1-s/2})z_0))_{m\leq i\leq m+l-1}, (a_i(e(\frac{t-s/2}{1-s/2})z_0))_{m+l\leq i\leq d}\Big) & 1 - s/2 \leq t \leq 1 \end{cases}$$

It is straightforward to check from the formulas given above that H is continuous; that $H(0,t)$ is the original loop; and that $H(1,t)$ is a concatenation of the loops described in the statement. It therefore suffices to show that $H(s,t) \in Y_d$ for all $(s,t) \in [0,1] \times [0,1]$.

It is easy to verify from Hypothesis 1.1 that the c_i's take distinct values on $B_{r'}^*$ for $i \in J$. It follows that the ith and jth coordinates of $H(s,t)$ are different for all (s,t) if $i, j \in J$ or if $i, j \notin J$. Now it clearly suffices to show that the ith coordinate of $H(s,t)$, which we denote by $H(s,t)_i$, satisfies $|H(s,t)_i - w| < \mu$ if and only if $i \in J$. Choose some $j \in J$ and some $i \notin J$, and let I be the unique subinterval containing i such that $(I,1) \in \mathcal{I}$. Now it can be directly verified from the formulas that $|H(s,t)_j - w| = |a_j(e(u)z_0) - w| < \mu$, and that meanwhile, $|H(s,t)_i - a_i(0)| = |a_i(e(u)z_0) - a_i(0)| < \mu$, where $u = 1$ if $0 \leq t \leq s/2$ and $u = \frac{t-s/2}{1-s/2}$ otherwise. Therefore, we have $|H(s,t)_j - w| < \mu$, and since $|a_i(0) - w| \geq 2\mu$, we have $|H(s,t)_i(0) - w| \geq \mu$. Since $j \in J$ and $i \notin J$ were chosen arbitrarily, we get the desired statement.

□

Lemma 2.7. *Let $p_1, ..., p_d$ be power series which converge on B_ν, for some $\nu > 0$. Define $\mathcal{I}$ for these power series as above, and assume that $(J,1)$ is in $\mathcal{I}$. Choose $z_0 \in B_\nu^*$, and let $p_i' = p_i$ for $i \notin J$ and $p_i' = p_i(0) + z_0 x^{-1}(p_i - p_i(0))$ for $i \in J$. Then there exists an $\nu' > 0$ which is independent of z_0, such that for all $z \in B_{\nu'}^*$ and $i = 1, ..., d$ we have $|p_i'(z) - p_i'(0)| < \frac{1}{2}\min_{p_j'(0) \neq p_i'(0)} |p_i'(0) - p_j'(0)|$.*

Proof. Fix the notation $p_i = \sum_{k=1}^{\infty} b_{i,k} x^k$ for $i = 1, ..., d$. Fix an index i, and let j be such that $p_j'(0) \neq p_i'(0)$.

If $i, j \notin J$, then $p_i' = p_i$ and $p_j' = p_j$ are defined independently of z_0. Since $p_i(z) - p_i(0)$ has no constant term, it follows that ν' can be chosen (independently of z_0) to be small enough that for $z \in B_{\nu'}^*$, we have

$$|p_i(z) - p_i(0)| < \tfrac{1}{2}|p_i(0) - p_j(0)| \neq 0. \tag{7}$$

If $i \notin J$ and $j \in J$, then $|p_i'(0) - p_j'(0)| = |p_i(0) - p_j(0) - z_0 b_{j,1}|$. Note that one can choose ν' to be small enough that $|p_i(0) - p_j(0) - z_0 b_{j,1}|$ is arbitrarily close to $|p_i(0) - p_j(0)| \neq 0$. Since $p_i'(z) - p_i'(0)$ has no constant term, it follows that ν' can be chosen (independently of z_0) to be small enough so that for all $z \in B_{\nu'}^*$, we have

$$|p_i'(z) - p_i'(0)| < \tfrac{1}{2}|p_i'(0) - p_j'(0)|. \tag{8}$$

If $i \in J$ and $j \notin J$, then $|p_i'(0) - p_j'(0)| = |p_i(0) + z_0 b_{i1} - p_j(0)|$, and ν' can be chosen (independently of z_0) so that $|p_i'(0) - p_j'(0)|$ is arbitrarily close to $|p_i(0) - p_j(0)| \neq 0$. The proof then follows as in the previous case.

If $i, j \in J$, then $p_i(0) = p_j(0)$, and therefore $|p_i'(0) - p_j'(0)| = |p_i(0) + z_0 b_{i,1} - p_j(0) - z_0 b_{j,1}| = |z_0||b_{i,1} - b_{j,1}|$. Meanwhile, $|p_i'(z) - p_i'(0)| = |z_0||(p_i - b_{i,0} - b_{i,1}z)/z|$. Therefore, the inequality $|p_i'(z) - p_i'(0)| < \frac{1}{2}|p_i'(0) - p_j'(0)|$ simplifies to

$$|(p_i - b_{i,0} - b_{i,1}z)/z| < \tfrac{1}{2}|b_{i,1} - b_{j,1}| \neq 0. \tag{9}$$

Since $(a_i - a_{i,0} - a_{i,1}z)/z$ is a power series with no constant term, again ν' can be chosen (independently of z_0) so that (9) holds for all $z \in B_{\nu'}^*$.

□

We are now ready to finish the proof of the main theorem of this section.

Proof. (of Theorem 2.3)

If $\mathcal{I}$ is empty, then $a_i(0) \neq a_j(0)$ for $i \neq j$. In this case, $\mathcal{F} \to B_\varepsilon$ is clearly a trivial fiber bundle and the action of δ is trivial. We therefore assume that $\mathcal{I}$ is not empty, so $N \geq 1$.

Let η, r and z_0 be as in Proposition 2.1. Choose an interval $J = \{m, ..., m+l-1\} \subseteq \{1, ..., d\}$ such that $(J,1) \in \mathcal{I}$. Now define power series $a_i' \in \mathbb{C}[[x]]$ by setting $a_i' = a_i$ for $i \in \{1, ..., m-1, m+l, ..., d\}$ and $a_i' = w + z_0 c_i$, where w and c_i are defined as in the statement of Lemma 2.6. Note that $a_i'(z_0) = a_i(z_0)$ for $1 \leq i \leq d$. Define $\mathcal{I}'$ for the power series a_i' in the same way that $\mathcal{I}$ was defined for the power series a_i. For any $(I', n') \in \mathcal{I}$, let $\lambda'_{I',n'} : [0,1] \to Y_d$ be the path defined with respect to the a_i''s analogously to how the $\lambda_{I,n}$'s were defined with respect to the a_i's. Note that $(J',1)$ is in $\mathcal{I}'$ if and only if $(J,2)$ is in $\mathcal{I}$ and that we have $\lambda'_{J',1} = \lambda_{J,2}$. The rest of the argument follows inductively from using Lemma 2.6 along with Lemma 4.3 below and choosing r to be small enough at each step. The fact that these choices of r are independent of z_0 follows immediately from Lemma 2.7.

$\square$

3. Proof of the comparison theorem

We first observe that the arithmetic part of the statement of Theorem 1.2(a) is an immediate corollary of the ideas in [Kis00]. In fact, slightly more is true.

Proposition 3.1. *Theorem 1.2(a) holds for $\rho_{\mathrm{alg}}^{(p')}$. Furthermore, if $R = \mathbb{C}[[x]]$, and q is any prime, then $\rho_{\mathrm{alg}}^{(q')}$ factors through $G_K^{(q')}$.*

Proof. The statement for $\rho_{\mathrm{alg}}^{(p')}$ follows from [Kis00, Corollary 1.16], which is stated in terms of outer Galois actions but easily implies the statement for full Galois actions.

In order to prove the statement for $\rho_{\mathrm{alg}}^{(q')}$ when $K = \mathbb{C}((x))$, we first observe that the statement of [Sza09, Corollary 5.5.8], which gives a condition for the injectivity of maps of fundamental groups, works for any Galois category (it is only stated there for the Galois category of finite étale covers). Therefore, it suffices in this case to prove that given any finite prime-to-p group G, every G-Galois cover of $\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}$ has a model defined over a prime-to-p extension of K. This follows from Proposition 4.8 (or from Remark 4.9) below, and so we are done.

$\square$

We will now justify a series of simplified assumptions. We will finally assume that $R = \mathbb{C}[[x]]$, in which case $p = 0$ and therefore Theorem 1.2(a) holds trivially.

3.1. Reduction to the case that $P = \infty$ and $\alpha_i \in R$.

It is clear that the basepoint P can be chosen arbitrarily, and so it suffices to prove the following.

Lemma 3.2. *It suffices to prove the statement of Theorem 1.2 in the case the $\alpha_i \in R$ for $1 \le i \le d$.*

Proof. We first observe that every K-automorphism ϕ of $\mathbb{P}^1_K$ induces an isomorphism

$$\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, \phi^{-1}(\infty)) \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\phi(\alpha_1), ..., \phi(\alpha_d)\}, \infty)$$

which respects the action of G_K.

Choose $\beta \in R^\times$ such that $\alpha_i - \beta \in R^\times$ for $1 \le i \le d$. (This is always possible because the residue field of R is infinite.) Then it is easy to check that the automorphism ϕ given by $z \mapsto z/(z - \beta)$ respects the intersection pairing. Thus, we may move any of the α_i's away from infinity, so assume that $\alpha_i \in K$ for $1 \le i \le d$.

Note that we may apply some power of the automorphism $\phi : z \mapsto \pi z$ to move the α_i's to elements of R, and that in general, this ϕ will change the intersection behavior of the α_i's. Let $E'_{i,j}$ be the intersection index of $\pi\alpha_i$ with $\pi\alpha_j$. Let $a'_1, ..., a'_d$ be elements of $\mathbb{C}[[x]]$ with intersection indices $E'_{i,j}$, and assume without loss of generality that the a'_i's as well as the a'_i's satisfy Hypothesis 1.1 for ε and that z_0 is chosen so that Theorem 2.3 will hold for both the a_i's and the a'_i's. By Remark 2.4(a), we may assume without loss of generality that $a'_i(z_0) = a_i(z_0)$ for $1 \le i \le d$. Define the action $(\bar{\rho}'_{\mathrm{top}})^{(p')}$ for the a'_i's in the same way that $\bar{\rho}_{\mathrm{top}}^{(p')}$ was defined for the a_i's. Then it follows from our observation at the beginning of the proof together with the statement of Theorem 1.2 for the case of points in R implies that the action $(\bar{\rho}'_{\mathrm{top}})^{(p')}$ factors through $G_K^{(p')}$ and is isomorphic to $\bar{\rho}_{\mathrm{alg}}^{(p')}$. It will therefore suffice to show that the outer monodromy actions $\bar{\rho}_{\mathrm{top}}$ and $\bar{\rho}'_{\mathrm{top}}$ are isomorphic.

Define $\mathcal{I}$ as in §2, and define $\mathcal{I}'$ analogously for the power series a'_i; similarly, define the loops $\gamma_{I,n}$ for $(I, n) \in \mathcal{I}$ as in §2 and define the loops $\gamma'_{I',n'}$ for $(I', n') \in \mathcal{I}$ analogously. Then it is clear that there is a bijection $\Phi : \mathcal{I} \to \mathcal{I}'$ given by $(I, n) \mapsto (I, n + 1)$ if $I \subseteq \{1, ..., l\}$, $(I, n) \mapsto (I, n - 1)$

if $I \subseteq \{l+1, ..., d\}$ and $n \geq 2$, and $(\{l+1, ..., d\}, 1) \mapsto (\{1, ..., l\}, 1)$. Note moreover that for each $(I, n) \in \mathcal{I}$, the loop $\mathrm{im}(\gamma_{I,n})$ separates the subset of points $\{a_i(z_0)\}_{i \in I}$ from its complement in $\{a_i(z_0)\}_{i=1}^{d}$ in $\mathbb{P}^1_{\mathbb{C}}$ and that the same statement holds for each $(I', n') \in \mathcal{I}'$. It is easy to see from this that the loop $\gamma_{I,n}$ is homotopic to the the loop $\gamma'_{\Phi((I,n))}$ on $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \setminus \{a_1(z_0), ..., a_d(z_0)\}$. Therefore the Dehn twists $D_{I,n}$ and $D_{\Phi((I,n))}$ act on $\pi_1(\mathcal{F}_{z_0}, \infty)$ in the same way up to inner automorphism. Now it follows from the description of the monodromy action in Theorem 2.3 that the actions $\bar{\rho}_{\mathrm{top}}$ and $\bar{\rho}'_{\mathrm{top}}$ are isomorphic, and we are done.

□

3.2. **Reduction to the case that R is countable.** We will assume for the remainder of this paper that $P = \infty$ and that $\alpha_i \in R$ for $1 \leq i \leq d$.

Lemma 3.3. *The sequence of morphisms $\mathbb{P}^1_{\bar{K}} \setminus \{\alpha_1, ..., \alpha_d\} \to \mathbb{P}^1_K \setminus \{\alpha_1, ..., \alpha_d\} \to \mathrm{Spec}(K)$ induces a split short exact sequence*

$$1 \to \pi_1^{\text{ét}}(\mathbb{P}_{\bar{K}} \setminus \{\alpha_1, ..., \alpha_d\}, P)^{(p')} \to \pi_1^{\text{ét}}(\mathbb{P}^1_K \setminus \{\alpha_1, ..., \alpha_d\}, P)^{(p')} \to G_K^{(p')} \to 1. \tag{10}$$

Furthermore, if $p = 0$, then for any prime q, we have

$$1 \to \pi_1^{\text{ét}}(\mathbb{P}_{\bar{K}} \setminus \{\alpha_1, ..., \alpha_d\}, P)^{(q')} \to \pi_1^{\text{ét}}(\mathbb{P}^1_K \setminus \{\alpha_1, ..., \alpha_d\}, P)^{(q')} \to G_K^{(q')} \to 1. \tag{11}$$

Proof. Exactness on the right is immediate from [RZ10, Proposition 3.2.5]. Exactness on the left follows from Proposition 3.1, using[And74, Proposition 4].

□

Lemma 3.4. *There is a strictly Henselian DVR R' with the following properties:*

a) R' is the completion of a countable subring of R with uniformizer π and which contains $\alpha_1, ..., \alpha_d$; and

b) letting K' denote the fraction field of R' and $(\rho'_{\mathrm{alg}})^{(p')} : G_{K'}^{(p')} \to \mathrm{Aut}(\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}'} \setminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')})$ be the Galois action analogous to $\rho_{\mathrm{alg}}^{(p')}$, there are isomorphisms $G_K^{(p')} \xrightarrow{\sim} G_{K'}^{(p')}$ and $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \setminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')} \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}'} \setminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')}$ inducing an isomorphism of the actions $\rho_{\mathrm{alg}}^{(p')}$ and $(\rho'_{\mathrm{alg}})^{(p')}$.

Proof. Let R'' be the integral closure of $R \cap \mathbb{Q}(\pi, \alpha_1, ..., \alpha_d)$ in R, and let K'' denote the fraction field of R''. Then R'' contains π and we have $\pi R \cap R'' = \pi R''$ because $R'' \subset R$ is an inclusion of integrally closed domains. The strict Henselization of the localization of R'' at $\pi R''$ is countable by construction; let R' be its completion. Then clearly R' satisfies the properties stated in (i). By Lemma 3.3, there are split short exact sequences of prime-to-p quotients of étale fundamental groups associated to the projective line minus the points $\alpha_1, ..., \alpha_d$ over the schemes $\mathrm{Spec}(K)$, $\mathrm{Spec}(K'')$, and $\mathrm{Spec}(K')$. We consider the commutative diagram below, where the rows are these split short exact sequences and the vertical arrows are induced by the field inclusions $\overline{K''} \subseteq \bar{K}$ and $\overline{K''} \subset \overline{K'}$.

(12)

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \setminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')} \arrow[r] \arrow[d] & \pi_1^{\text{ét}}(\mathbb{P}^1_K \setminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')} \arrow[r] \arrow[d] & G_K^{(p')} \arrow[r] \arrow[l, bend right] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(\mathbb{P}_{\overline{K''}} \setminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')} \arrow[r] & \pi_1^{\text{ét}}(\mathbb{P}^1_{K''} \setminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')} \arrow[r] & G_{K''}^{(p')} \arrow[r] \arrow[l, bend right] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{K'}} \setminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')} \arrow[r] \arrow[u] & \pi_1^{\text{ét}}(\mathbb{P}^1_{K'} \setminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')} \arrow[r] \arrow[u] & G_{K'}^{(p')} \arrow[r] \arrow[l, bend right] \arrow[u] & 1
\end{tikzcd}$$

Clearly the vertical arrows on the left are isomorphisms since they are induced by inclusions of algebraically closed fields. Note that it follows from a special case of Abhyankar's Lemma that the prime-to-p absolute Galois group of any strictly Henselian field of residue characteristic p is isomorphic to $\widehat{\mathbb{Z}}^{(p')}$. Therefore, the vertical arrows on the right are also isomorphisms. Then the vertical arrows in the middle are also isomorphisms by the Five Lemma. Since $\rho_{\mathrm{alg}}^{(p')}$ and $(\rho'_{\mathrm{alg}})^{(p')}$ are induced by the splittings of the top and bottom rows respectively, we are done.

□

3.3. **Reduction to the case** $R = \mathbb{C}[[x]]$**.** In order to reduce to the case of $R = \mathbb{C}[[x]]$, we will construct a ring (which we will call S) lying inside $\mathbb{C}[[x]]$ which has K as a quotient. This general strategy was inspired by a similar one used in a different context in [Oda95, §5].

We fix, once and for all, an embedding $R \hookrightarrow \mathbb{C}$. Let $S = R[[x]][\frac{1}{x}]$, and write F for its fraction field. The embedding $R \hookrightarrow \mathbb{C}$ determines an inclusion of F into the field $\mathbb{C}((x))$; let $\bar{F}$ be the algebraic closure of F inside $\overline{\mathbb{C}((x))}$. It is easy to verify that there is a unique R-algebra surjection $\psi : S \twoheadrightarrow K$, continuous in the x-adic topology, which sends x to π.

We now construct elements $\tilde{\alpha}_i \in R[x] \subset S$ with the same intersection behavior as the α_i's and satisfying $\psi(\tilde{\alpha}_i) = \alpha$. We observe that for any elements $\tilde{\alpha}_1, ..., \tilde{\alpha}_d \in S$ with $\psi(\tilde{\alpha}_i) = \alpha_i$ for $1 \leq i \leq d$, we have $e_{i,j} \leq E_{i,j}$ for $1 \leq i < j \leq d$, where $e_{i,j}$ is the x-adic valuation of $\tilde{\alpha}_i - \tilde{\alpha}_j$. Given such elements $\tilde{\alpha}_i \in R[x]$, we make adjustments to these polynomials in order to ensure that $e_{i,j} = E_{i,j}$ for $1 \leq i < j \leq d$, in the following manner. Suppose that $m := e_{1,2} < E_{1,2}$. Let $b \in R$ denote the coefficient of x^m in $\tilde{\alpha}_2 - \tilde{\alpha}_1$; it is clear from the fact that π^{m+1} divides $\alpha_2 - \alpha_1$ that b is divisible by π. Then add the element $\pi^{-1}bx^m(x - \pi) \in \ker(\psi)$ to $\tilde{\alpha}_2$, so that we still have $\psi(\tilde{\alpha}_2) = \alpha_2$ but now $m + 1 \leq e_{1,2} \leq E_{1,2}$. After repeating this process a finite number of times, we get $e_{1,2} = E_{1,2}$. Therefore, we may start by setting $\tilde{\alpha}_i = \alpha_i \in R \subset R[x]$ for $1 \leq i \leq d$, and then for j running through $\{2, ..., d\}$, and for i running through $\{1, ..., j - 1\}$, follow this procedure to ensure that $e_{i,j} = E_{i,j}$. It is easy to verify that this gives us polynomials $\tilde{\alpha}_i \in R[x]$ with the desired properties.

Lemma 3.5. *The divisor on the surface* $\mathbb{P}^1_S$ *given by the formal sum* $\sum_{i=1}^d (\tilde{\alpha}_i)$ *is étale over* $\mathrm{Spec}(S)$.

Proof. It suffices to show that the map $\bigcup_{i=1}^d (\tilde{\alpha}_i) \to \mathrm{Spec}(S)$ is unramified, or equivalently, that for $i \neq j$ and for each prime $\mathfrak{p}$ of S, the images of $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ modulo $\mathfrak{p}$ are distinct. We claim that any prime of S is either (π) or of the form (g), where $g \in R[x]$ is an irreducible monic polynomial all of whose non-leading coefficients lie in (π) (we call any monic polynomial with this property a *distinguished polynomial*). Let $\mathfrak{p}$ be any prime ideal of S different from (π). The p-adic Weierstrass preparation theorem says that each element of $R[[x]]$ can be written as the product $u\pi^m g$, where $u \in R[[x]]^\times$ is a unit, $m \geq 0$ is an integer, and g is a distinguished polynomial. Therefore, the prime $\mathfrak{p} \neq (\pi)$ of S is generated by elements $g_1, \pi^{m_2}g_2, ..., \pi^{m_r}g_r$, where $m_l \geq 0$ is an integer for $2 \leq l \leq r$ and $g_l \in R[x]$ is distinguished for $1 \leq l \leq r$. One checks using Gauss' Lemma that the greatest common divisor g of the g_l's is also a distinguished polynomial in $R[x]$. Clearly $\mathfrak{p}$ contains $\pi^m g$, where m is the maximum of the m_l's. Since $\mathfrak{p}$ is prime, we have $\pi \in \mathfrak{p}$ or $g \in \mathfrak{p}$. If $\pi \in \mathfrak{p}$, then since π divides the nonleading terms of the polynomial g, some power of x is contained in $\mathfrak{p}$, which contradicts the fact that x is a unit in S. It follows that $g \in \mathfrak{p}$ and in fact that $\mathfrak{p} = (g)$ with g irreducible, proving the above claim.

It now suffices to show that for $i \neq j$, the polynomials $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ are distinct modulo (π) and modulo (g) for any irreducible distinguished $g \in R[x] \smallsetminus \{x\}$. Let $m = e_{i,j} = E_{i,j}$, so that the mth coefficients $b_{i,m}$ and $b_{j,m}$ of $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ are distinct. Then, modulo π, we have $\psi(b_{j,m} - b_{i,m}) \equiv \psi(x^{-m}(\tilde{\alpha}_j - \tilde{\alpha}_i)) = \pi^{-m}(\alpha_j - \alpha_i) \in R^\times$, so we have $b_{j,m} - b_{i,m} \in R^\times$. In particular, this shows that $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ are distinct modulo (π). Now clearly, $S/(g) = K(\beta)$ where $\beta \in \bar{K}$ is a root of the polynomial g; note that β has positive valuation $s > 0$ due to the fact that g is distinguished. The images of $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ are given by the polynomials $\tilde{\alpha}_i(\beta)$ and $\tilde{\alpha}_j(\beta)$ in $K(\beta)$. Now $\tilde{\alpha}_i(\beta) - \tilde{\alpha}_j(\beta)$ can be written as a polynomial in β whose lowest-order term is $(b_{i,m} - b_{j,m})\beta^m$, which has valuation ms and is therefore nonzero. Thus, we have $\tilde{\alpha}_i(\beta) \neq \tilde{\alpha}_j(\beta)$, as desired.

□

We will now show that the prime-to-p Galois action associated to our scheme over $\mathbb{C}((x))$ is isomorphic to the one associated to our scheme over K.

Proposition 3.6. *With the above assumptions and notation, define* $\rho_0^{(p')} : G_{\mathbb{C}((x))} \to \mathrm{Aut}(\pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{C}((x))}} \smallsetminus \{\tilde{\alpha}_1, ..., \tilde{\alpha}_d\}, \infty)^{(p')})$ *to be the action induced by the action of* $G_{\mathbb{C}((x))}$ *on* $\pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{C}((x))}} \smallsetminus \{\tilde{\alpha}_1, ..., \tilde{\alpha}_d\}, \infty)$ *determined by the splitting of the fundamental short exact sequence*

$$1 \to \pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{C}((x))}} \smallsetminus \{\tilde{\alpha}_1, ..., \tilde{\alpha}_d\}, \infty) \to \pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{C}((x))} \smallsetminus \{\tilde{\alpha}_1, ..., \tilde{\alpha}_d\}, \infty) \to G_{\mathbb{C}((x))} \to 1 \tag{13}$$

induced by the point at infinity. Then $\rho_0^{(p')}$ *factors through* $G^{(p')}_{\mathbb{C}((x))}$*, and we have isomorphisms* $G_K^{(p')} \xrightarrow{\sim} G^{(p')}_{\mathbb{C}((x))}$ *and* $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')} \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{C}((x))}} \smallsetminus \{\tilde{\alpha}_1, ..., \tilde{\alpha}_d\}, \infty)^{(p')}$ *inducing an isomorphism of the actions* $\rho_{\text{alg}}^{(p')}$ *and* $\rho_0^{(p')}$.

Proof. By Proposition 3.1 applied to the fundamental short exact sequences associated to the families over both K and $\mathbb{C}((x))$, we have that $\rho_0^{(p')}$ and $\rho_{\text{alg}}^{(p')}$ factor through $G^{(p')}_{\mathbb{C}((x))}$ and $G_K^{(p')}$ respectively. By Lemma 3.3, taking prime-to-p quotients of the terms in the fundamental short exact sequences corresponding to each yields split short exact sequences of prime-to-p quotients inducing $\rho_0^{(p')}$ and $\rho_{\text{alg}}^{(p')}$ respectively. Therefore, it suffices to show that these two split short exact sequences of prime-to-p quotients are isomorphic. We will do so by constructing intermediate short exact sequences of prime-to-p quotients of étale fundamental groups, associated to a family of S-schemes, which are isomorphic to both.

Consider the family $\mathfrak{F}_S := \mathbb{P}^1_S \smallsetminus \{\tilde{\alpha}_1, ..., \tilde{\alpha}_d\} \to \mathrm{Spec}(S)$. We write $\mathfrak{F}_F$, $\mathfrak{F}_{\mathbb{C}((x))}$, and $\mathfrak{F}_K$ for the base changes of $\mathfrak{F}_S$ with respect to the inclusions $S \hookrightarrow F$ and $S \hookrightarrow \mathbb{C}((x))$ and the surjection $\psi : S \twoheadrightarrow K$ respectively, and we write $\mathfrak{F}_{\bar{F}}$, etc. for their geometric fibers. Note in particular that $\mathfrak{F}_K = \mathbb{P}^1_K \smallsetminus \{\alpha_1, ..., \alpha_d\}$. Let $\bar{\eta}$ be a generic geometric point over the generic point $\eta \in \mathrm{Spec}(S)$, and let $\bar{s}$ be a geometric point over the closed point $s := (x - \pi) \in \mathrm{Spec}(S)$. Since the divisor given by the points $\tilde{\alpha}_i \in \mathbb{P}^1_S$ is étale over $\mathrm{Spec}(S)$ by Lemma 3.5, we can apply [Gro71, Proposition XIII.4.3 and Exemples XIII.4.4], which shows that the sequence of morphisms $\mathfrak{F}_{\bar{K}} \to \mathfrak{F}_S \to \mathrm{Spec}(S)$ along with the section at infinity induces a split short exact sequence

$$1 \to \pi_1^{\text{ét}}(\mathfrak{F}_{\bar{K}}, \infty)^{(p')} \to \pi_1^{\text{ét}}(\mathfrak{F}_S, \infty_{\bar{s}})' \to \pi_1^{\text{ét}}(\mathrm{Spec}(S), \bar{s}) \to 1. \tag{14}$$

Here $\pi_1^{\text{ét}}(\mathfrak{F}_S, \infty)'$ is the quotient of $\pi_1^{\text{ét}}(\mathfrak{F}_S, \infty)$ constructed in [Gro71, §XIII.4]. In fact, the generalized version of Abhyankar's Lemma appearing as [GM71, Theorem 2.3.2], along with the fact that covers of $\mathrm{Spec}(S)$ of degree divisible by p are not separable, implies that the only étale covers of $\mathrm{Spec}(S)$ are induced by adjoining prime-to-p roots of x to S. Thus, we have $\pi_1^{\text{ét}}(\mathrm{Spec}(S), \bar{s}) = \pi_1^{\text{ét}}(\mathrm{Spec}(S), \bar{s})^{(p')} \cong \hat{\mathbb{Z}}^{(p')}$. It follows that $\pi_1^{\text{ét}}(\mathfrak{F}_S, \infty)' = \pi_1^{\text{ét}}(\mathfrak{F}_S, \infty)^{(p')}$, so (14) is actually a short exact sequence of prime-to-p quotients of étale fundamental groups.

Similarly, we have a split short exact sequence

$$1 \to \pi_1^{\text{ét}}(\mathfrak{F}_{\bar{F}}, \infty)^{(p')} \to \pi_1^{\text{ét}}(\mathfrak{F}_S, \infty_{\bar{\eta}})^{(p')} \to \pi_1^{\text{ét}}(\mathrm{Spec}(S), \bar{\eta})^{(p')} \to 1. \tag{15}$$

Choose compatible change-of basepoint isomorphisms $\pi_1^{\text{ét}}(\mathfrak{F}_S, \infty_{\bar{\eta}})^{(p')} \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathfrak{F}_S, \infty_{\bar{s}})^{(p')}$ and $\pi_1^{\text{ét}}(\mathrm{Spec}(S), \bar{\eta})^{(p')} \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathrm{Spec}(S), \bar{s})^{(p')}$. Since the divisor given by the points $\tilde{\alpha}_i \in \mathbb{P}^1_S$ is étale over $\mathrm{Spec}(S)$ by Lemma 3.5, we can apply a variant of Grothendieck's Specialization Theorem ([OV00, Théorème 4.4]) to obtain an isomorphism $\mathrm{sp} : \pi_1^{\text{ét}}(\mathfrak{F}_{\bar{F}}, \infty) \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathfrak{F}_{\bar{K}}, \infty)$. It is easy to check that sp commutes with the horizontal arrows in (15) and (14) and the change-of-basepoint isomorphisms. It follows that the sequence in (15) is isomorphic as a split short exact sequence to the one in (14).

Now these isomorphic exact sequences induced by the family over S fit into the commutative diagram below.

(16)
$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\text{ét}}(\mathfrak{F}_{\bar{K}}, \infty)^{(p')} \arrow[r] \arrow[d, equal] & \pi_1(\mathfrak{F}_K, \infty)^{(p')} \arrow[r] \arrow[d] & G_K^{(p')} \arrow[r] \arrow[l, bend right] \arrow[d, "\wr"] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(\mathfrak{F}_{\bar{K}}, \infty)^{(p')} \arrow[r] & \pi_1^{\text{ét}}(\mathfrak{F}_S, \infty_{\bar{s}})^{(p')} \arrow[r] & \pi_1^{\text{ét}}(\mathrm{Spec}(S), \bar{s})^{(p')} \arrow[r] \arrow[l, bend right] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(\mathfrak{F}_{\bar{F}}, \infty)^{(p')} \arrow[r] \arrow[u, "\text{sp}\wr"'] & \pi_1^{\text{ét}}(\mathfrak{F}_S, \infty_{\bar{\eta}})^{(p')} \arrow[r] \arrow[u, "\wr"'] & \pi_1^{\text{ét}}(\mathrm{Spec}(S), \bar{\eta})^{(p')} \arrow[r] \arrow[l, bend right] \arrow[u, "\wr"'] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(\mathfrak{F}_{\overline{\mathbb{C}((x))}}, \infty)^{(p')} \arrow[r] \arrow[u, "\wr"'] & \pi_1^{\text{ét}}(\mathfrak{F}_{\mathbb{C}((x))}, \infty)^{(p')} \arrow[r] \arrow[u] & G_{\mathbb{C}((x))}^{(p')} \arrow[r] \arrow[l, bend right] \arrow[u, "\wr"'] & 1
\end{tikzcd}$$

Here the vertical maps from the terms in the first row to those in the second row are induced by the surjection $\psi : S \twoheadrightarrow K$, and the vertical maps from the terms in the last row to those in the third row are induced by the inclusion $S \hookrightarrow \overline{\mathbb{C}((x))}$ given above. The bottom vertical arrows on the left is an isomorphism because $\bar{F} \hookrightarrow \overline{\mathbb{C}((x))}$ is an inclusion of algebraically closed fields, and the top and bottom vertical arrows on the right are isomorphisms because all of these groups are isomorphic to $\hat{\mathbb{Z}}^{(p')}$, as noted above. Therefore, the split short exact sequences at the top and bottom are isomorphic. Since the actions $\rho_{\text{alg}}^{(p')}$ and $\rho_0^{(p')}$ are induced by these sequences, we get the claimed isomorphism between $\rho_{\text{alg}}^{(p')}$ and $\rho_0^{(p')}$.

$\square$

Corollary 3.7. *In order to prove Theorem 1.2, it suffices to prove the statement for $R = \mathbb{C}[[x]]$, with the assumption that the elements α_i are polynomials.*

Proof. Since $\rho_0^{(p')}$ factors through $G_{\mathbb{C}((x))}^{(p')}$ by Proposition 3.3, it follows from the statement of Theorem 1.2 over $\mathbb{C}((x))$ that $\rho_{\text{top}}^{(p')}$ factors through $\pi_1(B_\varepsilon^*, z_0)^{(p')}$ and that the actions $\rho_0^{(p')}$ and $\rho_{\text{top}}^{(p')}$ are isomorphic. The desired statement then follows from Proposition 3.6.

$\square$

3.4. **Proof of Theorem 1.2(b).** By what has been shown in the above subsections, along with Remark 2.4(a), we may assume with loss of generality that $R = \mathbb{C}[[x]]$ and that $a_i = \alpha_i \in \mathbb{C}[x]$ for $1 \leq i \leq d$. The statement of Theorem 1.2(b) now results from the following proposition. (For a sketch of an alternative approach, see the end of [Wew02, §2.3]. Note that Remark 2.13 in the same paper does not imply a description of the non-equicharacteristic inertia action, because [Wew02, Theorem 2.12] and its proof are insufficient for determining this action.)

Proposition 3.8. *Theorem 1.2(b) holds when $R = \mathbb{C}[[x]]$ and $\alpha_i = a_i \in \mathbb{C}[x]$ for $1 \leq i \leq d$.*

Proof. Let $X = \mathbb{A}^1_{\mathbb{C}} \smallsetminus (\{0\} \cup \{z \mid a_i(z) = a_j(z) \text{ for some } i \neq j\})$. Let $\mathcal{F}' \to X$ be the family given by

$$\mathcal{F}' = (\mathbb{P}^1_{\mathbb{C}} \times X) \smallsetminus \bigcup_{i=1}^{d} \{(a_i(z), z) \mid z \in X\}.$$

Then $(\mathcal{F}')^{\text{an}} \to X^{\text{an}}$ is clearly a fiber bundle which has $\mathcal{F} \to B_\varepsilon^*$ as a sub-bundle. Fix an algebraic closure $\overline{\mathbb{C}((x))}$ of $\mathbb{C}((x))$, and let $\bar{\eta} : \mathrm{Spec}(\overline{\mathbb{C}((x))}) \to X$ be the corresponding geometric point lying over the generic point of X. The induced morphism $\widehat{\mathbb{Z}} \cong G_{\mathbb{C}((x))} \to \pi_1^{\text{ét}}(X, \bar{\eta})$ is an injection since every nontrivial element of $\pi_1^{\text{ét}}(X, \bar{\eta})$ has infinite order. Similarly, the inclusion $B_\varepsilon^* \subset X^{\text{an}}$ induces a map $\widehat{\mathbb{Z}} \cong \widehat{\pi}_1(B_\varepsilon^*, z_0) \to \pi_1^{\text{ét}}(X, z_0)$ which is also an injection.

A path between z_0 and $\bar{\eta}$ induces an isomorphism of $\pi_1^{\text{ét}}(X, z_0)$ with $\pi_1^{\text{ét}}(X, \bar{\eta})$. We remark that such a path can be chosen so that the images of both $G_{\mathbb{C}((x))}$ and $\widehat{\pi}_1(B_\varepsilon^*, z_0)$ in $\pi_1^{\text{ét}}(X, z_0)$ are equal. Indeed, if $Y = X \cup \{0\} \subset \mathbb{C}$, then the kernel of $\pi_1^{\text{ét}}(X, z_0) \to \pi_1^{\text{ét}}(Y, z_0)$ is topologically generated by the conjugates of the image of $\widehat{\pi}_1(B_\varepsilon^*, z_0)$; and for any path, the induced image of $G_{\mathbb{C}((x))}$ in $\pi_1^{\text{ét}}(X, z_0)$ is contained in this kernel. Since varying the path conjugates the image of $G_{\mathbb{C}((x))}$, there exists a path so that its image is contained in the image of $\widehat{\pi}_1(B_\varepsilon^*, z_0)$. To show that for such a path the images are equal, it suffices to check their restrictions with respect to a cover of X of the form $y^n = x$, where it is clear.

Let $\infty_{\bar{\eta}}$ in $\mathcal{F}'_{\bar{\eta}}$ be the $\overline{\mathbb{C}((x))}$-point lying over the point at infinity. Then it is easy to see that there exists a path between ∞_{z_0} and $\infty_{\bar{\eta}}$ that induces an isomorphism $\pi_1^{\text{ét}}(\mathcal{F}', \infty_{\bar{\eta}}) \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathcal{F}', \infty_{z_0})$ that commutes with the isomorphism $\pi_1^{\text{ét}}(X, \infty_{\bar{\eta}}) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \infty_{z_0})$ above, as well as with the infinity sections. By [Gro71, Corollaire X.2.4], there exists a "specialization map"

$$\mathrm{sp} : \pi_1^{\text{ét}}(\mathcal{F}'_{\bar{\eta}}, \infty_{\bar{\eta}}) \twoheadrightarrow \pi_1^{\text{ét}}(\mathcal{F}'_{z_0}, \infty),$$

which Grothendieck's Specialization Theorem ([Gro71, Corollaire X.3.9]) says is an isomorphism. We obtain the following commutative diagram of short exact sequences of groups.

$$\begin{tikzcd}
1 \arrow[r] & \widehat{\pi}_1(\mathcal{F}_{z_0}, \infty_{z_0}) \arrow[r] \arrow[d, equal] & \widehat{\pi}_1(\mathcal{F}, \infty_{z_0}) \arrow[r] \arrow[d, hook] & \widehat{\pi}_1(B_\varepsilon^*, z_0) \arrow[r] \arrow[l, bend right] \arrow[d, hook] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(\mathcal{F}'_{z_0}, \infty_{z_0}) \arrow[r] & \pi_1^{\text{ét}}(\mathcal{F}', \infty_{z_0}) \arrow[r] & \pi_1^{\text{ét}}(X, z_0) \arrow[r] \arrow[l, bend right] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(\mathcal{F}'_{\bar{\eta}}, \infty_{\bar{\eta}}) \arrow[r] \arrow[u, "\mathrm{sp}\wr"] & \pi_1^{\text{ét}}(\mathcal{F}', \infty_{\bar{\eta}}) \arrow[r] \arrow[u, "\wr"] & \pi_1^{\text{ét}}(X, \bar{\eta}) \arrow[r] \arrow[l, bend right] \arrow[u, "\wr"] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(\mathcal{F}'_{\bar{\eta}}, \infty_{\bar{\eta}}) \arrow[r] \arrow[u, "\wr"] & \pi_1^{\text{ét}}(\mathcal{F}'_{\mathbb{C}((x))}, \infty_{\mathbb{C}((x))}) \arrow[r] \arrow[u, hook] & G_{\mathbb{C}((x))} \arrow[r] \arrow[l, bend right] \arrow[u, hook] & 1
\end{tikzcd} \tag{17}$$

It is easy to check from the construction of sp in [Gro71, Exposé X] that the middle two squares commute. The top row is obtained by applying the right-exact functor of taking profinite completions to the short exact sequence (2) of (topological) fundamental groups. The fact that the bottom three rows are short exact sequences follows from [Gro71, Proposition XIII.4.3 and Exemples XIII.4.4]. The left-exactness of the top row follows immediately from a diagram chase; it is obvious that ρ_{top} is induced by the splitting of this row. The top and bottom maps shown in the middle row are injections by the Five Lemma.

The commutativity of the above diagram shows that the top and bottom rows are isomorphic as split short exact sequences and that therefore the induced actions are isomorphic, as desired.

$\square$

Remark 3.9. We observe that it is implicit in the proofs of Propositions 3.6 and 3.8 that if the α_i's satisfy the additional hypothesis in the second statement of Theorem 1.2(b), we have $\widehat{\pi}_1(\mathcal{F}, \infty_{z_0})^{(p')} \cong \pi_1^{\text{ét}}(\mathbb{P}^1_K \smallsetminus \{\alpha_1, ..., \alpha_d\}, P)^{(p')}$.

Remark 3.10. We make the identification $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')}$ with $\widehat{\pi}_1((\mathbb{P}^1_{\mathbb{C}})^{\text{an}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')}$ via Riemann's Existence Theorem and the embedding $\bar{K} \hookrightarrow \mathbb{C}$ fixed in the discussion leading up to Lemma 3.5. Then it is clear from the proofs of Propositions 3.8 and 3.6 that we can say a bit more about the isomorphism $\widehat{\pi}_1(\mathcal{F}_{z_0}, \infty_{z_0})^{(p')} \xrightarrow{\sim} \widehat{\pi}_1((\mathbb{P}^1_{\mathbb{C}})^{\text{an}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')}$ given in the statement of Theorem 1.2 (which is essentially a composition of maps coming from specializations and inclusions of algebraically closed fields). Namely, this isomorphism takes the topological fundamental group of $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{a_1(z_0), ..., a_d(z_0)\}$ to the topological fundamental group of $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{\alpha_1, ..., \alpha_d\}$ for $1 \leq i \leq d$. Moreover, if $\sigma \in \widehat{\pi}_1(\mathcal{F}_{z_0}, \infty_{z_0})^{(p')}$ is the image of some element of $G_{\mathbb{C}(x)}$ which lies in the conjugacy class of generators of inertia at the prime $(x - a_i(z_0)) \in \mathrm{Spec}(\mathbb{C}[x])$, then this isomorphism takes σ to the image in $\widehat{\pi}_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, \infty)^{(p')}$ of an element $G_{\mathbb{C}(x)}$ which lies in the conjugacy class of generators of inertia at the prime $(x - \alpha_i) \in \mathrm{Spec}(\mathbb{C}[x])$. If such a σ actually lies in the topological fundamental group $\pi_1(\mathcal{F}_{z_0}, \infty_{z_0})$, then σ may be viewed as a counterclockwise loop in $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{a_1(z_0), ..., a_d(z_0)\}$ winding around only the missing point $a_i(z_0)$, and the analogous statement is true for the image of σ in $\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, \infty)$ (see [Vol96, Remark 5.10]). In this way, we see that the above isomorphism of prime-to-p étale fundamental groups takes images of loops around the point $a_i(z_0)$ to images of loops around the point α_i for each i.

4. Applications and examples

In this section we provide an algorithm for explicit computations of the monodromy actions that are the subject of Theorem 1.2. This is done by first giving an algorithm for explicit computations for the topological monodromy and then using Theorem 1.2 to give explicit description of the arithmetic monodromy action.

4.1. **Explicit algorithm for computing the topological monodromy action.** Let $a_1, ..., a_d$ be power series in $\mathbb{C}[[x]]$ satisfying Hypothesis 1.1. In this section we will introduce a set of generators of $\pi_1(\mathcal{F}_{z_0}, \infty_{z_0})$ for which we give an explicit description of the monodromy action by $\pi_1(B^*_\varepsilon, z_0)$. Our approach is similar to the one taken in [Yu96, §6].

We reprise all of the notation used in §2.1 and fix $T_0 = (a_1(z_0), ..., a_d(z_0)) \in Y_d$. Given an integer $d \geq 2$, the *full braid group on d strands*, denoted B_d, is defined to be the group of homotopy classes of paths in the ordered configuration space Y_d of the affine line which begin at the point $(1, ..., d) \in Y_d$ and end at a point in Y_d given by a permutation of the ordered set $(1, ..., d)$. The *pure braid group on d strands*, denoted P_d, is defined to be the fundamental group $\pi_1(Y_d, (1, ..., d))$; we view it as a normal subgroup of B_d in the obvious way.

There is a well-known description of B_d as an abstract group, given by generators $\beta_1, ..., \beta_{d-1}$ (where each β_i is the braid rotating the i^{th} and $(i+1)^{\mathrm{st}}$ points in a counterclockwise semicircular motion) and relations

$$\begin{cases} \beta_i\beta_j = \beta_j\beta_i, & |i-j| \geq 2 \\ \beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}, & 1 \leq i \leq d-2. \end{cases} \tag{18}$$

Recall the fiber bundle $Y_{d+1} \to Y_d$ defined by "forgetting" the $(d+1)$st point. This fiber bundle induces a short exact sequence of fundamental groups, which is split by the section $s : Y_d \to Y_{d+1}$ given by $T \mapsto 1 + \max_{z \in T} |z|$. Meanwhile, we recall the fiber bundle $\bar{Y}_{d+1} \to Y_d$ of punctured Riemann spheres and note that the section defined above deforms to the infinity section $T \mapsto \infty_T$. We therefore have the following commutative diagram, where the top and bottom rows are the short exact sequences associated to these two fiber bundles.

(19)
$$\begin{tikzcd} 1 \arrow[r] & \pi_1((\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\}, d+1) \arrow[r] \arrow[d] & \pi_1(Y_{d+1}, (1, ..., d+1)) = P_{d+1} \arrow[r] \arrow[d] & \pi_1(Y_d, (1, ..., d)) = P_d \arrow[r] \arrow[l, bend right] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \pi_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\}, d+1) \arrow[r] & \pi_1(\bar{Y}_{d+1}, (1, ..., d+1)) \arrow[r] & \pi_1(Y_d, (1, ..., d)) = P_d \arrow[r] \arrow[l, bend right] & 1 \end{tikzcd}$$

Assume Hypothesis 2.5, and assume that we have chosen η and r as in Theorem 2.3. Fix a path in $\bar{Y}_{d+1}$ from $\infty_{T_0} = (a_1(z_0), ..., a_d(z_0), \infty) \in \bar{Y}_{d+1}$ to $(1, ..., d, d+1)$ so that each of the loops

$\gamma_{I,n}$ is taken to a loop in $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\}$ which is homotopic to a circular loop surrounding the subinterval $I = \{m, ..., m+l-1\} \subseteq \{1, ..., d\}$. From now on, we identify $\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\}, d+1)$ with $\pi_1(\mathcal{F}_{z_0}, \infty_{z_0})$ via this path.

We will specify a generating set for the fundamental group $\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\}, d+1)$ by specifying one for $\pi_1((\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\}, d+1)$ as follows. For $i = 1, ..., d$ let x_i be the loop on $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$ based at the point $d+1$ going above the points $i+1, ..., d$ and wrapping counterclockwise around only the point i. It is clear from the top short exact sequence in (19) that we may view these x_i's as elements of $P_{d+1} \lhd B_{d+1}$. It is easy to verify that as an element of B_{d+1}, each generator x_i can be expressed as $(\beta_d \cdots \beta_{i+1})\beta_i^2(\beta_d \cdots \beta_{i+1})^{-1}$.

We see from the diagram in (19) that the action of P_d on $\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\}, d+1)$ induced by the splitting of the bottom short exact sequence is given by the restriction of the conjugation action of $B_d \subset B_{d+1}$ on the subgroup $\pi_1((\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\}, d+1) \subset B_{d+1}$ (which is normalized by B_d). The following lemma, which describes this conjugation action, follows from straightforward calculations using the braid relations given in (18).

Lemma 4.1. *The elements x_i, viewed as braids in B_{d+1}, behave under conjugation by the braids $\beta_i \in B_d \subset B_{d+1}$ as follows.*

$$\begin{cases} \beta_i^{-1} x_{i+1} \beta_i = x_i, & \\ \beta_i^{-1} x_i \beta_i = x_i x_{i+1} x_i^{-1}, & \\ \beta_i^{-1} x_j \beta_i = x_j, & j \neq i, i+1. \end{cases}$$

In order to compute the action of the Dehn twists on the x_i's, we need a couple more lemmas.

Lemma 4.2. *For any basepoint $T \in Y_d$, the representation $\rho : \pi_1(Y_d, T) \to \mathrm{Aut}(\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus T, \infty))$ factors through $\pi_0 \mathcal{Y}_d$ as the composition $\varphi \circ \partial$.*

Proof. To simplify notation, we will assume that $T = (1, ..., d)$. Let $\beta : [0,1] \to Y_d$ be a loop based at $(1, ..., d)$, which induces a loop on $\bar{Y}_{d+1}$ via the infinity section which we also denote by β. Let $\gamma : [0,1] \to (\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\} \hookrightarrow \bar{Y}_{d+1}$ be a loop based at ∞. We want to show that the concatenation of loops $\beta^{-1}\gamma\beta$ is a representative of the element $\varphi(\partial(\beta))([\gamma]) \in \pi_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\}, \infty)$. It is clear from the above discussion and the diagram in (19) that we may deform β and γ so that β is a loop $[0,1] \to Y_d \overset{s}{\hookrightarrow} \bar{Y}_{d+1}$ and γ is a loop on $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\}$ based at $d+1$. Then it suffices to show that the concatenation of loops given by $\beta^{-1}\gamma\beta$ is homotopic to the loop obtained by acting on γ by a representative of $\partial([\beta])$ in $\mathcal{Y}_d$.

From the discussion in §2.1, we see that the loops $\beta, \gamma : [0,1] \to Y_{d+1}$ lift via the fiber bundle $\epsilon : \mathcal{Y}_0 \to Y_{d+1}$ to paths $\tilde{\beta}, \tilde{\gamma} : [0,1] \to \mathcal{Y}_0$ starting at id $\in \mathcal{Y}_0$ and ending at representatives of $\partial([\beta]), \partial([\gamma]) \in \pi_0 \mathcal{Y}_d$ respectively. Therefore, $\beta^{-1}\gamma\beta$ lifts to the path $[0,1] \to \mathcal{Y}_0$ given by $t \mapsto \tilde{\beta}(t)^{-1}\tilde{\gamma}(t)\tilde{\beta}(t)$. We claim that this path is homotopic to the path $\tilde{\delta} : t \mapsto \tilde{\beta}(1)^{-1}\tilde{\gamma}(t)\tilde{\beta}(1)$. Indeed, there is a homotopy $[0,1] \times [0,1] \to \mathcal{Y}_0$ of paths starting at id and ending at $\tilde{\beta}(1)^{-1}\tilde{\gamma}(1)\tilde{\beta}(1)$ given by $(s,t) \mapsto \tilde{\beta}(t + s(1-t))^{-1}\tilde{\gamma}(t)\tilde{\beta}(t + s(1-t))$ which deforms $\tilde{\beta}^{-1}\tilde{\gamma}\tilde{\beta}$ to $\tilde{\delta}$. Now since $\tilde{\beta}(1)$ fixes each of the points $1, ..., d+1$ while $\tilde{\gamma}(t)$ fixes each of the points $1, ..., d$ for all t, it is easy to check that $\tilde{\delta}(t)$ fixes each point $1, ..., d$ and takes $d+1$ to $(\tilde{\beta}(1) \circ \gamma)(t)$ for all t. It follows that $\tilde{\delta} : [0,1] \to \mathcal{Y}_0$ is the (unique) lifting of the loop obtained by acting on γ by $\tilde{\beta}(1) \in \mathcal{Y}_d$, which is a representative of $\partial([\beta])$. This implies the desired homotopy of loops on $\bar{Y}_{d+1}$.

□

Lemma 4.3. *For choices of η, r and z_0 as in Proposition 2.1, the loop $\lambda_{I,n}$ defined at the beginning of §2.3 acts on $\pi_1((\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus T_0, \infty)$ as $\varphi(D_{I,n})$.*

Proof. By Lemma 4.2, it is enough to show that $\partial([\lambda_{I,n}]) = D_{I,n} \in \pi_0 \mathcal{Y}_d$. Choose some real number $\xi > 0$ small enough so that $\xi < r^{n-1}\eta$ and $\{z \in \mathbb{C} \mid r^{n-1}\eta - \xi < |z - w| < r^{n-1}\eta + \xi\} \subset (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$ does not contain any of the points $a_i(z_0)$. Let $\tilde{\lambda}_{I,n} : [0,1] \times \mathcal{Y}_0 \to (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus T_0$ be the homotopy such that for all $t \in [0,1]$, $\tilde{\lambda}_{I,n}(t) : (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \to (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$ acts on $\{z \in \mathbb{C} \mid |z - w_{I,n}| > r^{n-1}\eta + \xi\}$ as the identity, on $\{z \in \mathbb{C} \mid |z - w_{I,n}| < r^{n-1}\eta - \xi\}$ as $z \mapsto e(t)z$, and on $\{z \in \mathbb{C} \mid r^{n-1}\eta - \xi < |z - w_{I,n}| < r^{n-1}\eta + \xi\}$ by fixing the outer rim and twisting the inner rim counterclockwise by an angle of $2\pi t$. Then clearly $\tilde{\lambda}_{I,n}(0)$ is the identity $\mathrm{id} \in \mathcal{Y}_0$, and $\tilde{\lambda}_{I,n}(t)$ agrees with $\lambda_{I,n}(t)$ on the points $a_1(z_0), ..., a_d(z_0)$ for all $t \in [0,1]$. It follows from the construction of ∂ given above that $\partial([\lambda_{I,n}])$ is represented by $\tilde{\lambda}_{I,n}(1)$ in $\pi_0\mathcal{Y}_d$. Since by definition, the Dehn twist $D_{I,n} \in \pi_0\mathcal{Y}_d$ is also represented by $\tilde{\lambda}_{I,n}(1)$, we are done.

□

Proposition 4.4. *For any $(I,n) \in \mathcal{I}$ with $I = \{m, ..., m+l-1\}$, the Dehn twist $D_{I,n} \in \pi_0\mathcal{Y}_d$ acts on $\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}} \smallsetminus \{1, ..., d\}\}, d+1)$ as*

$$\begin{cases} x_i \mapsto (x_m \cdots x_{m+l-1}) x_i (x_m \cdots x_{m+l-1})^{-1}, & i \in I \\ x_i \mapsto x_i, & i \notin I. \end{cases}$$

Proof. We first directly observe that the loop $\lambda_{I,n} \in P_d$ can be expressed in terms of the generators β_i of B_d as $(\beta_m \cdots \beta_{m+l-2})^l \in P_d$. Therefore Lemma 4.3 implies that $\rho((\beta_m \cdots \beta_{m+l-2})^l) = \varphi(D_{I,n})$, and so it will suffice to show, using Lemma 4.1, that $(\beta_m \cdots \beta_{m+l-2})^l$ acts on each x_i in the manner described in the above statement.

It is straightforward to check that the the element $\beta_m \cdots \beta_{m+l-2} \in B_d$ acts on the generators x_i by (right) conjugation as

$$x_m \mapsto (x_m \cdots x_{m+l-2}) x_{m+l-1} (x_m \cdots x_{m+l-2})^{-1} = (x_m \cdots x_{m+l-1}) x_{m+l-1} (x_m \cdots x_{m+l-1})^{-1},$$

as $x_i \mapsto x_{i-1}$ for $m+1 \leq i \leq m+l-1$, and as $x_i \mapsto x_i$ for $i \notin I$. We easily deduce from these observations that conjugation by $\beta_m \cdots \beta_{m+l-2}$ fixes the product $x_m \cdots x_{m+l-1}$. This, combined with our earlier observations, shows that $(\beta_m \cdots \beta_{m+l-2})^l$ conjugates each x_i by $(x_m \cdots x_{m+l-1})^{-1}$ for $i \in I$ while fixing each x_i for $i \notin I$.

□

Remark 4.5. In fact, it is not difficult to see from Lemma 4.2 and the formulas given in Lemma 4.1 that more generally, each element of $\pi_0\mathcal{Y}_d$ acts on each generator x_i by taking it to a conjugate of x_i.

4.2. **Explicit computations of prime-to-p étale fundamental groups.** Remark 3.9 now allows us to compute prime-to-p fundamental groups of the sort $\pi_1^{\text{ét}}(\mathbb{P}^1_K \smallsetminus \{\alpha_1, ..., \alpha_d\}, P)^{(p')}$ with explicit generators and relations. We give two basic examples below.

Example 4.6. Given any elements $a_1, ..., a_d \in \mathbb{Z}_p^{\mathrm{un}} \cup \{\infty\}$ algebraic over $\mathbb{Q}$ that are pairwise distinct modulo p, we have (for any basepoint P)

$$\pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{Q}_p^{\mathrm{un}}} \smallsetminus \{a_1, ..., a_d\}, P)^{(p')} \cong \widehat{\langle x_1, ..., x_d, \delta \mid x_1 \cdots x_d = 1, \{[\delta, x_i] = 1\}_{1 \leq i \leq d} \rangle}^{(p')}.$$

Example 4.7. Given any prime $p \geq 3$ and integer $m \geq 1$, we have (for any basepoint P)

$$\pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{Q}_p^{\mathrm{un}}} \smallsetminus \{0, p^m, 1, 2\}, P)^{(p')} \cong$$

$$\widehat{\langle x_1, x_2, x_3, x_4, \delta \mid x_1 \cdots x_4 = 1, \{\delta^{-1} x_i \delta = (x_1 x_2)^m x_i (x_1 x_2)^{-m}\}_{i=1,2}, \{[\delta, x_i] = 1\}_{i=3,4} \rangle}^{(p')}.$$

4.3. **The field of moduli of prime-to-p covers.** Recall that a given G-Galois branched cover of $\mathbb{P}^1_{\bar{K}}$ has a unique minimal field of definition as a Galois cover, namely its field of moduli over K (as a G-Galois cover). Indeed, the obstruction for the field of moduli to be a field of definition is contained in $H^2(K, Z(G_K))$. (See, for example, [Bel80, §1], [Dè90, Note below Theorem 1], [Dè95, §6.3].) Since K is a strictly Henselian field, this cohomology group vanishes.

Combining the explicit algorithm in Section 4.1 with Theorem 1.2 gives us the following corollary about the field of moduli of prime-to-p Galois branched covers of $\mathbb{P}^1_{\bar{K}}$.

Corollary 4.8. *Let G be a finite prime-to-p group, and let $Y \to \mathbb{P}^1_{\bar{K}}$ be a G-Galois branched cover which is ramified only over K-rational points. Then the degree of its field of moduli (as a G-Galois cover) over K divides the exponent of the quotient $G/Z(G)$.*

Proof. Let $\alpha_1, ..., \alpha_d$ be the K-points of $\mathbb{P}^1_K$ over which the map $Y \to \mathbb{P}^1_{\bar{K}}$ is ramified, so that this cover corresponds to a surjection $\phi : \pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, P) \twoheadrightarrow G$ (where P is some basepoint). Let $a_1, ..., a_d \in \mathbb{C}[[x]]$ be power series satisfying Hypothesis 1.1 which have the same intersection behavior as the α_i's. Let $x_1, ..., x_d$ be the set of generators of $\pi_1((\mathbb{P}^1_{\mathbb{C}})^{\text{an}} \smallsetminus T_0, \infty_{z_0})$ specified in §4.1, and let $\bar{x}_1, ..., \bar{x}_d$ be their images under the isomorphism from $\hat{\pi}_1((\mathbb{P}^1_{\mathbb{C}})^{\text{an}} \smallsetminus T_0, \infty)^{(p')}$ to $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\}, P)^{(p')}$ guaranteed by Theorem 1.2. It follows from Proposition 4.4 (or Remark 4.5), Theorem 2.3, and Theorem 1.2 that up to inner automorphism, a topological generator $\bar{\delta}$ of $G_K^{(p')}$ acts by taking each $\bar{x}_i$ to $\bar{y}_i^{-1}\bar{x}_i\bar{y}_i$ for some $\bar{y}_i \in \pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \smallsetminus \{\alpha_1, ..., \alpha_d\})^{(p')}$. Let N denote the exponent of $G/Z(G)$. Then $\bar{\delta}^N$ acts up to inner automorphism as $\bar{x}_i \mapsto \bar{y}_i^{-N}\bar{x}_i\bar{y}_i^N$, so for $1 \le i \le d$, we have that the elements $\phi(\bar{x}_i^{\bar{\delta}^N})$ are uniformly conjugate to the elements $\phi(\bar{y}_i)^{-N}\phi(\bar{x}_i)\phi(\bar{y}_i)^N = \phi(\bar{x}_i)$, where the last equality follows from the fact that the N^{th} power of any element of G lies in $Z(G)$. It follows that the field of moduli (as a G-Galois cover) of $Y \to \mathbb{P}^1_{\bar{K}}$ is contained in the fixed field of $\bar{\delta}^N$, which is $K(\pi^{1/N})$.

□

Remark 4.9.

a) With some work one can show that Théorèmes 3.2 and 3.7 in [Saï97], adapted to our situation, imply that $K(t^{1/M})$ is a field of definition of the cover (together with its Galois action), where M is the exponent of G. The corollary above is a strengthening of this result.

b) We observe that in the case of $K = \mathbb{C}((x))$, the proof of the above corollary relies only on Proposition 3.8 rather than the full statement of Theorem 1.2. This case can therefore be used to prove Proposition 3.1 above.

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