POLARIZATIONS ON COMPLEX ABELIAN VARIETIES

by Jeff Yelton

Let $X \cong V/\Lambda$ be a complex abelian variety, D a divisor on X, and $\mathcal{L} = \mathcal{L}(D)$ the associated line bundle. We have seen that associated to each line bundle $\mathcal{L}$ is a (normalized) theta function $\theta : V \to \mathbb{C}$ with the property that

$$\theta(z+\lambda) = e^{\pi H(z,\lambda)+\frac{1}{2}\pi H(\lambda,\lambda)+2\pi i K(\lambda)}\theta(z) \quad \forall z \in V, \lambda \in \Lambda$$

where H is a Hermitian form on V, $E = \mathrm{Im}(H)$, and $K : \Lambda \to \mathbb{R}$ is a linear functional satisfying

$$K(\lambda_1+\lambda_2) - K(\lambda_1) - K(\lambda_2) \equiv \frac{1}{2}E(\lambda_1,\lambda_2) \pmod{\mathbb{Z}} \quad \forall \lambda_1, \lambda_2 \in \Lambda.$$

Conversely, for every such H and K, there is a unique divisor class in $\mathrm{Pic}^0(X)$ (i.e. a line bundle $\mathcal{L}(H,K)$ unique up to isomorphism). Each H determines an element of the Neron-Severi group $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$, so the elements of $\mathrm{Pic}^0(X)$ correspond to $H \equiv 0$ and K satisfying

$$K(\lambda_1,\lambda_2) - K(\lambda_1) - K(\lambda_2) \equiv 0 \pmod{\mathbb{Z}} \quad \forall \lambda_1, \lambda_2 \in \Lambda.$$

In other words, for $\mathcal{L}$ a divisor class viewed as a line bundle (defined up to isomorphism) in $\mathrm{Pic}^0(X)$, $\chi_{\mathcal{L}} : \Lambda \to \mathbb{C}, \lambda \mapsto e^{2\pi i K(\lambda)}$ is a character on Λ, and it completely determines the divisor class $\mathcal{L}$ in $\mathrm{Pic}^0(X)$. (In fact, $\mathrm{Pic}^0(X)$ may be identified with the group of characters on Λ.) On the other hand, Hermitian forms H which are positive definite correspond to ample line bundles $\mathcal{L} \in \mathrm{Pic}(X)$.

Now recall the definition of polarization:

Definition 0.1. *For any abelian variety X, a **polarization** is a homomorphism from X to $\mathrm{Pic}^0(X)$ of the form*

$$\phi_{\mathcal{L}} : x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

for an ample line bundle $\mathcal{L}$ of X.

For a complex abelian variety X, we want to identify $\mathrm{Pic}^0(X)$ with $X^\vee$, the dual of X, and see explicitly how any polarization $\phi_{\mathcal{L}}$ is an isogeny of X onto $X^\vee$.

Let $\mathcal{L}$ be an ample line bundle on X inducing the map $\phi_{\mathcal{L}} : X \to \mathrm{Pic}^0(X), x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$, with associated H and K, so that we can write $\mathcal{L}$ as $\mathcal{L}(H, K)$. For a point $x \in X$, we want to find the Hermitian form t_x^*H and linear functional t_x^*K associated to $t_x^*\mathcal{L}$. Let $a \in V$ be any point which projects down to $x \in X \cong V/\Lambda$. Then the functional relation of the pullback $\theta' := t_a^*\theta$ associated to $t_x^*\mathcal{L}$ can be written as

$$\theta'(z+\lambda) = e^{\pi H(z+a,\lambda)+\frac{1}{2}H(\lambda,\lambda)+2\pi i K(\lambda)}\theta'(z) \quad \forall z \in V, \lambda \in \Lambda.$$

We may replace $\theta'(z)$ by $e^{\pi H(z,a)}\theta'(z)$ since $e^{-\pi H(z,a)}$ is a non-vanishing holomorphic function on V. This adds a factor of $e^{-\pi H(\lambda,a)}$ to the functional equation above, so that now we have

$$\begin{aligned}\theta'(z+\lambda) &= e^{\pi H(z+a,\lambda)+\frac{1}{2}H(\lambda,\lambda)+2\pi i K(\lambda)-H(\lambda,a)}\theta'(z)\\ &= e^{\pi H(a,\lambda)-\pi H(\lambda,a)} \cdot e^{\pi H(z,\lambda)+\frac{1}{2}H(\lambda,\lambda)+2\pi i K(\lambda)}\theta'(z)\\ &= e^{2\pi i E(a,\lambda)} \cdot e^{\pi H(z,\lambda)+\frac{1}{2}H(\lambda,\lambda)+2\pi i K(\lambda)}\theta'(z) \quad \forall z \in V, \lambda \in \Lambda.\end{aligned}$$

Therefore we see that $t_x^*H = H$ and $t_x^*K(\lambda) = E(a,\lambda)+K(\lambda)$. It follows that $\phi_{\mathcal{L}}(x) = t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$ has associated Hermitian form $t_x^*H - H = 0$ and linear functional $t_x^*K - K = E(a,\lambda)$. Therefore, the element $t_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \in \mathrm{Pic}^0(X)$ is uniquely characterized by the function $\chi_{\mathcal{L},x} : \lambda \mapsto e^{2\pi i E(a,\lambda)}$, which is in fact a character on Λ. (Note that this is defined independently of the choice of $a \in V$ lying above $x \in X$.)

This leads to the following proposition:

Proposition 0.2. *Let X be a complex abelian variety and let $\mathcal{L} = \mathcal{L}(H, K)$ be an ample line bundle on X (H is a positive definite Hermitian form and $E = \mathrm{Im}(H)$ is the associated non-degenerate Riemann form). Then the map*

$$\phi_{\mathcal{L}} : X \to x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

as described above is a surjective homomorphism of abstract groups.

Proof: We have characterized this map as $x \mapsto \chi_{\mathcal{L},x}$ as defined above. It is obvious from the construction that $\chi_{\mathcal{L},x_1+x_2} = \chi_{\mathcal{L},x_1}\chi_{\mathcal{L},x_2}$, so we have a homomorphism of abstract groups. Next we want to show that it is surjective. By above discussion, we may view $\mathrm{Pic}^0(X)$ as the group of characters on Λ. Any such character is of the form $\lambda \mapsto e^{2\pi i K(\lambda)}$, where K is some $\mathbb{R}$-linear functional on V. But since $E : V \times V \to \mathbb{R}$ is a non-degenerate pairing, every

$\mathbb{R}$-linear functional on V is of the form $z \mapsto E(a, z)$ for some $a \in V$. Defining K as $\lambda \mapsto E(a, \lambda)$ shows that our character in $\mathrm{Pic}^0(X)$ is $\mathcal{L}' = \mathcal{L}'(0, K)$. But by the above discussion, $\mathcal{L}' \cong t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$, where $x \in X$ is the image of a modulo Λ. Therefore, every element in $\mathrm{Pic}^0(X)$ is $\phi_{\mathcal{L}}(x)$ for some $x \in X$.

Proposition 0.3. *Choose a basis $\{v_1, v_2, ..., v_{2g}\}$ of V. Using the notation of Proposition 1, the kernel of $\phi_{\mathcal{L}}$ is a finite subgroup of X of order* $\det(E)$, *where E is considered as the matrix whose (i, j)th entry is $E(v_i, v_j)$.*

Remark 0.4. *By Frobenius, there exists a symplectic basis $\{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g\}$ of the free $\mathbb{Z}$-module Λ for the alternating form E. This symplectic basis has the property that $E(\alpha_i, \alpha_j) = E(\beta_i, \beta_j) = 0$ for $1 \leq i, j \leq g$, and $E(\alpha_i, \beta_i) = d_i$ for $1 \leq i \leq g$ for integers $d_1|d_2|...|d_g$. Then E, when expressed as a matrix with respect to this symplectic basis, has integer determinant $(d_1 d_2 ... d_g)^2$. Since $\mathcal{L}$ is ample, E is non-degenerate, so $d_i \neq 0$ for $1 \leq i \leq g$, and therefore,* $\det(E)$ *(which does not depend on the choice of basis) must be a positive integer.*

Proof. The kernel of $\phi_{\mathcal{L}}$ is precisely those $x \in X$ such that the character $\chi_{\mathcal{L},x}$ is trivial – that is, $E(a, \lambda) \in \mathbb{Z}$ for all $\lambda \in \Lambda$ (and a choice of $a \in V$ whose image modulo Λ is x). Write a as a vector $\langle a_1, ..., a_g, b_1, ..., b_g \rangle$ with respect to the symplectic basis from Remark 3. Then $E(a, \alpha_i) = -d_i b_i$ and $E(a, \beta_i) = d_i a_i$. The condition that $E(a, \lambda) \in \mathbb{Z}$ for all $\lambda \in \Lambda$, therefore, is equivalent to $-d_1 b_1, ..., -d_g b_g, d_1 a_1, ..., d_g a_g \in \mathbb{Z}$. This is the case if and only if a is an element of the free $\mathbb{Z}$-module generated by $\{\alpha_1/d_1, ..., \alpha_g/d_g, \beta_1/d_1, ..., \beta_g/d_g\}$, which is a full lattice containing Λ as a sublattice of index $(d_1 d_2 ... d_g)^2 = \det(E)$. This lattice modulo Λ is a finite subgroup of X of order $\det(E)$, thus proving the proposition.

□

Our final goal is to describe $\mathrm{Pic}^0(X)$ as a complex abelian variety ($X^\vee$, the dual of X), such that $\phi_{\mathcal{L}}$ is an isogeny. In order to do this, we use the definition of a complex abelian variety as a complex torus V^*/Λ^* equipped with a non-degenerate Riemann form H^*. We have already shown that $\mathrm{Pic}^0(X)$ can be identified with the group of characters on Λ. Each character can be expressed as $\lambda \mapsto e^{2\pi i K(\lambda)}$ for K an $\mathbb{R}$-linear functional of V. Let $\xi : V \to \mathbb{C}$ be defined as $\xi(a) = K(ia) - iK(a)$. Then one checks that ξ is a $\mathbb{C}$-antilinear homomorphism with $K = -\mathrm{Im}(\xi)$. In this way, we see that there is a surjective homomophism

$$\mathrm{Hom}_{\mathbb{C}-\mathrm{anti}}(V, \mathbb{C}) \twoheadrightarrow \mathrm{Pic}^0(X),\ \xi \mapsto [\lambda \mapsto e^{-2\pi i \mathrm{Im}(\xi(\lambda))}].$$

The kernel of this homomorphism is

$$\Lambda^* := \{\xi \in V^* \mid \mathrm{Im}(\xi(\lambda)) \in \mathbb{Z}\ \forall \lambda \in \Lambda\}.$$

It is easy to check that V^* is a g-dimensional vector space over $\mathbb{C}$ and that Λ^* is a full lattice in V^*. Therefore, $\mathrm{Pic}^0(X) \cong V^*/\Lambda^*$ is a complex torus.

The homomorphism $\phi_{\mathcal{L}} : V/\Lambda \to V^*/\Lambda^*$ can be described as follows. Each element of V^*/Λ^* corresponds to a character on Λ given by $\lambda \mapsto e^{2\pi i E(a,\lambda)}$ for some $a \in V$. This functional $E(a, \cdot)$ "comes from" the $\mathbb{C}$-antilinear map $H(a, \cdot) = E(ia, \cdot) + iE(a, \cdot)$ in V^*. Therefore, define the homomorphism $\tilde{\phi_{\mathcal{L}}} : V \to V^*$ as $a \mapsto H(a, \cdot)$. One can check that $\tilde{\phi_{\mathcal{L}}}(\Lambda) \leq \Lambda^*$ and that therefore, $\tilde{\phi_{\mathcal{L}}}$ induces a homomorphism on the quotients, which is in fact $\phi_{\mathcal{L}} : V/\Lambda \to V^*/\Lambda^*$. Now by the non-degeneracy of H, $\tilde{\phi_{\mathcal{L}}}$ is actually an isomorphism. Therefore, the finiteness of the kernel of $\phi_{\mathcal{L}}$ implies that $\tilde{\phi_{\mathcal{L}}}(\Lambda) \leq \Lambda^*$ is a sublattice of finite index (in fact, the index is equal to $\det(E)$).

Now define

$$H^* : V^* \times V^* \to \mathbb{C},\ H^*(\xi_1, \xi_2) = H(\tilde{\phi_{\mathcal{L}}}^{-1}(\xi_1), \tilde{\phi_{\mathcal{L}}}^{-1}(\xi_2)).$$

H^* is clearly a positive definite Hermitian form on V^*, but $\mathrm{Im}(H^*)$ may not take integer values on Λ^*. However, $\mathrm{Im}(H^*)$ certainly takes integer values on $\tilde{\phi_{\mathcal{L}}}(\Lambda)$, which, as previously stated, is a sublattice of finite index of Λ^*, so we may multiply H^* by a suitable integer to get a non-degenerate Riemann form on V^*.

In this way, we conclude that $\mathrm{Pic}^0(X)$ can be given the structure of a complex abelian variety $X^\vee$, the *dual abelian variety* of X. For each ample line bundle $\mathcal{L}$ on X, the polarization $\phi_{\mathcal{L}}$ is an isogeny from X to $X^\vee$.