Let $X \cong V/\Lambda$ be a complex abelian variety, $D$ a divisor on $X$, and $\mathcal{L} = \mathcal{L}(D)$ the associated line bundle. We have seen that associated to each line bundle $\mathcal{L}$ is a (normalized) theta function $\theta : V \to \mathbb{C}$ with the property that

$$\theta(z + \lambda) = e^{\pi H(z, \lambda) + \frac{1}{2} \pi H(\lambda, \lambda) + 2\pi i K(\lambda)} \theta(z) \quad \forall z \in V, \lambda \in \Lambda$$

where $H$ is a Hermitian form on $V$, $E = \text{Im}(H)$, and $K : \Lambda \to \mathbb{R}$ is a linear functional satisfying

$$K(\lambda_1 + \lambda_2) - K(\lambda_1) - K(\lambda_2) \equiv \frac{1}{2} E(\lambda_1, \lambda_2) \pmod{\mathbb{Z}} \quad \forall \lambda_1, \lambda_2 \in \Lambda.$$

Conversely, for every such $H$ and $K$, there is a unique divisor class in $\text{Pic}^0(X)$ (i.e. a line bundle $\mathcal{L}(H, K)$ unique up to isomorphism). Each $H$ determines an element of the Neron-Severi group $\text{Pic}(X)/\text{Pic}^0(X)$, so the elements of $\text{Pic}^0(X)$ correspond to $H \equiv 0$ and $K$ satisfying

$$K(\lambda_1, \lambda_2) - K(\lambda_1) - K(\lambda_2) \equiv 0 \pmod{\mathbb{Z}} \quad \forall \lambda_1, \lambda_2 \in \Lambda.$$

In other words, for $\mathcal{L}$ a divisor class viewed as a line bundle (defined up to isomorphism) in $\text{Pic}^0(X)$, $\chi_{\mathcal{L}} : \Lambda \to \mathbb{C}, \lambda \mapsto e^{2\pi i K(\lambda)}$ is a character on $\Lambda$, and it completely determines the divisor class $\mathcal{L}$ in $\text{Pic}^0(X)$. (In fact, $\text{Pic}^0(X)$ may be identified with the group of characters on $\Lambda$.) On the other hand, Hermitian forms $H$ which are positive definite correspond to ample line bundles $\mathcal{L} \in \text{Pic}(X)$.

Now recall the definition of polarization:

**Definition 0.1.** For any abelian variety $X$, a **polarization** is a homomorphism from $X$ to $\text{Pic}^0(X)$ of the form

$$\phi_{\mathcal{L}} : x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

for an ample line bundle $\mathcal{L}$ of $X$.

For a complex abelian variety $X$, we want to identify $\text{Pic}^0(X)$ with $X^\vee$, the dual of $X$, and see explicitly how any polarization $\phi_{\mathcal{L}}$ is an isogeny of $X$ onto $X^\vee$. 
Let $\mathcal{L}$ be an ample line bundle on $X$ inducing the map $\phi_{\mathcal{L}} : X \to \text{Pic}^0(X)$, $x \mapsto t^*_x \mathcal{L} \otimes \mathcal{L}^{-1}$, with associated $H$ and $K$, so that we can write $\mathcal{L}$ as $\mathcal{L}(H, K)$. For a point $x \in X$, we want to find the Hermitian form $t^*_x H$ and linear functional $t^*_x K$ associated to $t^*_x \mathcal{L}$. Let $a \in V$ be any point which projects down to $x \in X \cong V/\Lambda$. Then the functional relation of the pullback $\theta' := t^*_a \theta$ associated to $t^*_a \mathcal{L}$ can be written as

$$
\theta'(z + \lambda) = e^{\pi H(z+a,\lambda) + \frac{1}{2} H(\lambda,\lambda) + 2\pi i K(\lambda)} \theta'(z) \quad \forall z \in V, \lambda \in \Lambda.
$$

We may replace $\theta'(z)$ by $e^{\pi H(z,a)} \theta'(z)$ since $e^{-\pi H(z,a)}$ is a non-vanishing holomorphic function on $V$. This adds a factor of $e^{-\pi H(a,\lambda)}$ to the functional equation above, so that now we have

$$
\theta'(z + \lambda) = e^{\pi H(z+a,\lambda) + \frac{1}{2} H(\lambda,\lambda) + 2\pi i K(\lambda) - H(a,\lambda)} \theta'(z)
= e^{\pi H(a,\lambda) - \pi H(\lambda,\lambda)} \cdot e^{\pi H(z,\lambda) + \frac{1}{2} H(\lambda,\lambda) + 2\pi i K(\lambda)} \theta'(z)
= e^{2\pi i E(a,\lambda)} \cdot e^{\pi H(z,\lambda) + \frac{1}{2} H(\lambda,\lambda) + 2\pi i K(\lambda)} \theta'(z) \quad \forall z \in V, \lambda \in \Lambda.
$$

Therefore we see that $t^*_x H = H$ and $t^*_x K(\lambda) = E(a, \lambda) + K(\lambda)$. It follows that $\phi_{\mathcal{L}}(x) = t^*_x \mathcal{L} \otimes \mathcal{L}^{-1}$ has associated Hermitian form $t^*_x H - H = 0$ and linear functional $t^*_x K - K = E(a, \lambda)$. Therefore, the element $t^*_x \mathcal{L} \otimes \mathcal{L}^{-1} \in \text{Pic}^0(X)$ is uniquely characterized by the function $\chi_{\mathcal{L}, x} : \lambda \mapsto e^{2\pi i E(a,\lambda)}$, which is in fact a character on $\Lambda$. (Note that this is defined independently of the choice of $a \in V$ lying above $x \in X$.)

This leads to the following proposition:

**Proposition 0.2.** Let $X$ be a complex abelian variety and let $\mathcal{L} = \mathcal{L}(H, K)$ be an ample line bundle on $X$ ($H$ is a positive definite Hermitian form and $E = \text{Im}(H)$ is the associated non-degenerate Riemann form). Then the map

$$
\phi_{\mathcal{L}} : X \to x \mapsto t^*_x \mathcal{L} \otimes \mathcal{L}^{-1}
$$

as described above is a surjective homomorphism of abstract groups.

**Proof:** We have characterized this map as $x \mapsto \chi_{\mathcal{L}, x}$ as defined above. It is obvious from the construction that $\chi_{\mathcal{L}, x_1 + x_2} = \chi_{\mathcal{L}, x_1} \chi_{\mathcal{L}, x_2}$, so we have a homomorphism of abstract groups. Next we want to show that it is surjective. By above discussion, we may view $\text{Pic}^0(X)$ as the group of characters on $\Lambda$. Any such character is of the form $\lambda \mapsto e^{2\pi i K(\lambda)}$, where $K$ is some $\mathbb{R}$-linear functional on $V$. But since $E : V \times V \to \mathbb{R}$ is a non-degenerate pairing, every
\[ \mathbb{R}-\text{linear functional on } V \text{ is of the form } z \mapsto E(a, z) \text{ for some } a \in V. \]

Defining \( K \) as \( \lambda \mapsto E(a, \lambda) \) shows that our character in \( \text{Pic}^0(X) \) is \( \mathcal{L}' = \mathcal{L}'(0, K) \). But by the above discussion, \( \mathcal{L}' \cong t_x^* \mathcal{L} \otimes \mathcal{L}^{-1} \), where \( x \in X \) is the image of \( a \) modulo \( \Lambda \). Therefore, every element in \( \text{Pic}^0(X) \) is \( \phi_\mathcal{L}(x) \) for some \( x \in X \).

**Proposition 0.3.** Choose a basis \( \{v_1, v_2, \ldots, v_{2g}\} \) of \( V \). Using the notation of Proposition 1, the kernel of \( \phi_\mathcal{L} \) is a finite subgroup of \( X \) of order \( \det(E) \), where \( E \) is considered as the matrix whose \((i, j)\)th entry is \( E(v_i, v_j) \).

**Remark 0.4.** By Frobenius, there exists a symplectic basis \( \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\} \) of the free \( \mathbb{Z}\)-module \( \Lambda \) for the alternating form \( E \). This symplectic basis has the property that \( E(\alpha_i, \alpha_j) = E(\beta_i, \beta_j) = 0 \) for \( 1 \leq i, j \leq g \), and \( E(\alpha_i, \beta_i) = d_i \) for \( 1 \leq i \leq g \) for integers \( d_1 | d_2 | \ldots | d_g \). Then \( E \), when expressed as a matrix with respect to this symplectic basis, has integer determinant \( (d_1d_2\ldots d_g)^2 \). Since \( \mathcal{L} \) is ample, \( E \) is non-degenerate, so \( d_i \neq 0 \) for \( 1 \leq i \leq g \), and therefore, \( \det(E) \) (which does not depend on the choice of basis) must be a positive integer.

**Proof.** The kernel of \( \phi_\mathcal{L} \) is precisely those \( x \in X \) such that the character \( \chi_{\mathcal{L}, x} \) is trivial – that is, \( E(a, \lambda) \in \mathbb{Z} \) for all \( \lambda \in \Lambda \) (and a choice of \( a \in V \) whose image modulo \( \Lambda \) is \( x \)). Write \( a \) as a vector \( \langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle \) with respect to the symplectic basis from Remark 3. Then \( E(a, \alpha_i) = -d_i b_i \) and \( E(a, \beta_i) = d_i a_i \). The condition that \( E(a, \lambda) \in \mathbb{Z} \) for all \( \lambda \in \Lambda \), therefore, is equivalent to \( -d_1 b_1, \ldots, -d_g b_g, d_1 a_1, \ldots, d_g a_g \in \mathbb{Z} \). This is the case if and only if \( a \) is an element of the free \( \mathbb{Z}\)-module generated by \( \{\alpha_1/d_1, \ldots, \alpha_g/d_g, \beta_1/d_1, \ldots, \beta_g/d_g\} \), which is a full lattice containing \( \Lambda \) as a sublattice of index \( (d_1d_2\ldots d_g)^2 = \det(E) \). This lattice modulo \( \Lambda \) is a finite subgroup of \( X \) of order \( \det(E) \), thus proving the proposition.

\[ \square \]

Our final goal is to describe \( \text{Pic}^0(X) \) as a complex abelian variety \( (X^\vee, \text{the dual of } X) \), such that \( \phi_\mathcal{L} \) is an isogeny. In order to do this, we use the definition of a complex abelian variety as a complex torus \( V^*/\Lambda^* \) equipped with a non-degenerate Riemann form \( H^* \). We have already shown that \( \text{Pic}^0(X) \) can be identified with the group of characters on \( \Lambda \). Each character can be expressed as \( \lambda \mapsto e^{2\pi i K(\lambda)} \) for \( K \) an \( \mathbb{R}\)-linear functional of \( V \). Let \( \xi : V \to \mathbb{C} \) be defined as \( \xi(a) = K(ia) - iK(a) \). Then one checks that \( \xi \) is a \( \mathbb{C}\)-antilinear homomorphism with \( K = -\text{Im}(\xi) \). In this way, we see that there is a surjective homomorphism

\[ \text{Hom}_{\mathbb{C}-\text{anti}}(V, \mathbb{C}) \to \text{Pic}^0(X), \ \xi \mapsto \left[ \lambda \mapsto e^{-2\pi i \text{Im}(\xi(\lambda))} \right]. \]
The kernel of this homomorphism is
\[ \Lambda^* := \{ \xi \in V^* \mid \text{Im}(\xi(\lambda)) \in \mathbb{Z} \ \forall \lambda \in \Lambda \}. \]

It is easy to check that \( V^* \) is a \( g \)-dimensional vector space over \( \mathbb{C} \) and that \( \Lambda^* \) is a full lattice in \( V^* \). Therefore, \( \text{Pic}^0(X) \cong V^*/\Lambda^* \) is a complex torus.

The homomorphism \( \phi_L : V/\Lambda \to V^*/\Lambda^* \) can be described as follows. Each element of \( V^*/\Lambda^* \) corresponds to a character on \( \Lambda \) given by \( \lambda \mapsto e^{2\pi i E(a, \lambda)} \) for some \( a \in V \). This functional \( E(a, \cdot) \) “comes from” the \( \mathbb{C} \)-antilinear map \( H(a, \cdot) = E(ia, \cdot) + iE(a, \cdot) \) in \( V^* \). Therefore, define the homomorphism \( \tilde{\phi}_L : V \to V^* \) as \( a \mapsto H(a, \cdot) \). One can check that \( \tilde{\phi}_L(\Lambda) \leq \Lambda^* \) and that therefore, \( \tilde{\phi}_L \) induces a homomorphism on the quotients, which is in fact \( \phi_L : V/\Lambda \to V^*/\Lambda^* \). Now by the non-degeneracy of \( H \), \( \phi_L \) is actually an isomorphism. Therefore, the finiteness of the kernel of \( \phi_L \) implies that \( \tilde{\phi}_L(\Lambda) \leq \Lambda^* \) is a sublattice of finite index (in fact, the index is equal to \( \det(E) \)).

Now define
\[ H^* : V^* \times V^* \to \mathbb{C}, \ H^*(\xi_1, \xi_2) = H(\tilde{\phi}_L^{-1}(\xi_1), \tilde{\phi}_L^{-1}(\xi_2)). \]

\( H^* \) is clearly a positive definite Hermitian form on \( V^* \), but \( \text{Im}(H^*) \) may not take integer values on \( \Lambda^* \). However, \( \text{Im}(H^*) \) certainly takes integer values on \( \tilde{\phi}_L(\Lambda) \), which, as previously stated, is a sublattice of finite index of \( \Lambda^* \), so we may multiply \( H^* \) by a suitable integer to get a non-degenerate Riemann form on \( V^* \).

In this way, we conclude that \( \text{Pic}^0(X) \) can be given the structure of a complex abelian variety \( X^\vee \), the dual abelian variety of \( X \). For each ample line bundle \( L \) on \( X \), the polarization \( \phi_L \) is an isogeny from \( X \) to \( X^\vee \).