Abelian Varieties and basepoint freeness

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Abstract

The aim of these notes is to give a proof of the basepoint freness part of the Fujita's Conjecture for Complex Abelian Varieties, following [Mum], §2. Here we give a short introduction to the general Fujita's Conjecture (see [Laz1], §11.2).

Consider a smooth projective curve C of genus g, and suppose that D is a divisor of degree d on C. A classical (and elementary) theorem states that if $d \geq 2g$ then the complete linear system $|D|$ is basepoint free, and if $d \geq 2g+1$ then D is very ample. It is natural to ask whether this extends to smooth varieties of arbitrary dimension. Interestingly enough, the shape that such a statement might take came into focus only in the late 1980's. The starting point is to rephrase the result for curves without explicitly mentioning the genus g of C. Specifically, note that

$$\deg D \geq 2g \iff D \equiv K_C + L \,, \tag{0.1}$$

where $\equiv$ denotes the numerical equivalence, L is an ample divisor of degree ≥ 2 and similarly $\deg D \geq 2g+1$ if and only if D admits the same expression with $\deg L \geq 3$. This suggests that bundles of large degree on a curve generalize to adjoint divisors on a smooth projective variety X of arbitrary dimension, i.e., divisors of the form $K_X + L$ for a suitably positive divisor L on X. These considerations are made precise in a conjecture due to Fujita.

Conjecture 0.1 (Fujita's Conjecture)**.** *Let X be a smooth projective variety of dimension n, and A an ample divisor on X. Then:*

(i) If $m \geq n+1$ then $K_X + mA$ is basepoint free.

(ii) If $m \geq n+2$ then $K_X + mA$ is very ample.

As observed before, the conjecture is true if $\dim(X) = 1$. It has been veriefied also for $\dim(X) \leq 5$ in the case of global generation (see [Rei], [EL], [Kaw], [Hel], [YZ]). There are also partial results for

the case of very ampleness when X is a Calabi–Yau threefold (see [GP]), and more generally when X has a nef canonical bundle (see [MR]). For arbitrary dimension, there exist strong global generation statements due to Angehrn and Siu (see [AS]) and Heier (see [Hei]), whose bounds are nevertheless not linear in $\dim(X)$. While sharp for hyperplane bundles on projective spaces, Fujita's conjecture is very far from the truth in general. One notable class for which part (*ii*) of the Conjecture is true is that of Abelian Varieties: we know from Lefschetz theorem that if D is an ample divisor on an Abelian Variety X, then $3D$ is very ample (recall that the canonical divisor of an abelian variety is trivial, since the tangent bundle is trivial). We want to show in these notes that also part (*i*) of the conjecture is true for Abelian Varieties with much smaller multiples than the one expected: if D is an ample divisor on an Abelian Variety X, then $2D$ is base point free. Furthermore we will also show that in this case the morphism induced by the linear system $|2D|$ is finite.

Another important example is given by K3 surfaces (see [SD]): if D is an ample divisor on a K3 surface X, then $2D$ is basepoint free and $3D$ is very ample, so also in this case bounds are lower than expected by the Conjecture.

1 Some basic definitions and results on Complex Abelian Varieties

1.1 Divisors on Complex Abelian Varieties

First of all we recall the definition of Abelian Variety given at the beginning of the course (where not specified we refer to [Yel]).

Definition 1.1. A *group variety* over a field k is an algebraic variety A over k with the property that there is a group law on the set of points $A(k)$ such that group addition and inversion are given by morphisms $m : A \times A \to A$ and $i : A \to A$. An *abelian variety* is a group variety which is complete, i.e., for every algebraic variety X the projection $\mathrm{pr}_X : A \times X \to X$ is closed.

Remark 1.2. An abelian variety is smooth. Indeed, on any variety over k, there is a k-point x_0 at which the variety is smooth. If A is an abelian variety, for each $y \in A$, the translation-by-y map $t_y : A \to A$ given by $x \mapsto y + x$ is an invertible morphism from A to itself and therefore induces an isomorphism on the tangent spaces $(t_y)_* : T_{x_0}X \xrightarrow{\sim} T_{y+x_0}X$. Since every $a \in A$ is equal to $y + x_0$ for some y, the tangent spaces at all points of A are isomorphic to $T_{x_0}A$, and so A is smooth.

From now on, when not specified, we work with Complex Abelian Varieties, so we will take $k = \mathbb{C}$. We now recall the definition of Cartier and Weil divisors.

Definition 1.3. Let X be a connected complex manifold.

(i) The additive group $\mathrm{Div}(X)$ of "local function data" on X is defined as follows. A "local function datum" is given by $D = \{\{U_i\}_{i\in I}, \{f_i\}_{i\in I}\}$, where $\{U_i\}_{i\in I}$ is a finite open cover of X and, for each $i \in I$, f_i is a nonzero meromorphic function defined on the open subset $U_i \subset X$ such that for any $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, the quotient f_i/f_j (and f_j/f_i) is holomorphic and nonvanishing on $U_i \cap U_j$. We define the sum of two elements $D_1 = \{\{U_i\}_{i\in I}, \{f_i\}_{i\in I}\}$ and $D_2 = \{\{V_j\}_{j\in J}, \{g_j\}_{j\in J}\}$ to be $D_1 + D_2 := \{\{U_i \cap V_j\}_{(i,j)\in I\times J}, \{f_i g_j\}_{(i,j)\in I\times J}\}$.

(ii) A "local function datum" $\{\{U_i\}_{i\in I}, \{f_i\}_{i\in I}\}$ defined in this way is said to be *effective* if each f_i is holomorphic on U_i. It is said to be *trivial* if each f_i is both holomorphic and nonvanishing on U_i.

(iii) The group of Cartier divisors on X, denoted $\mathrm{Div}(X)$, is defined to be the group equivalence classes of "local function data" on X modulo trivial elements. If a divisor D is identified in this way with some $\{\{U_i\}_{i\in I}, \{f_i\}_{i\in I}\}$ for some open cover $\{U_i\}_{i\in I}$, then we say that D can be represented by this data.

(iv) A divisor $D \in \mathrm{Div}(X)$ is said to be *principal* if it can be represented by $\{\{X\}, \{f\}\}$ for some nonzero meromorphic function f on X. In this case we denote D by (f).

(v) Two Cartier divisors $D, D' \in \mathrm{Div}(X)$ are *linearly equivalent* if $D - D'$ is principal. In this case we write $D \sim D'$.

We observe that the Cartier divisor group is commutative and that the trivial divisor, which we denote by 0, is the identity element. The notion of effectiveness of Cartier divisors induces a partial ordering on the Cartier divisor group. We write $D \geq D'$ if $D - D'$ is effective.

Definition 1.4. Let X be a normal variety over a field k.

(i) A *prime divisor* on X is a closed subvariety of X of codimension 1. A *Weil divisor* is an element of the free abelian group $\mathrm{WDiv}(X)$ generated by prime divisors, i.e., $D \in \mathrm{WDiv}(X)$ is a formal finite sum $D = \sum d_Z Z$, where $d_Z \in \mathbb{Z}$ and Z is a prime divisor.

(ii) A Weil divisor $D = \sum d_Z Z$ is *effective* if every d_Z is nonnegative.

(iii) Denote by $K^*(X)$ the set of nonzero rational function on X. If $f \in K^*(X)$, the *Weil divisor asssociated to* f is $(f) := \sum \mathrm{ord}_Z f \cdot Z$, where the sum is over all the prime divisors of Y and $\mathrm{ord}_Z(f)$ is the order of f along Z (see [HAG], II, for more details).

(iv) A Weil Divisor $D \in \mathrm{WDiv}(X)$ is *principal* if $D = (f)$ for some $f \in K^*(X)$.

(v) Two Weil divisors $D, D' \in \mathrm{WDiv}(X)$ are *linearly equivalent* if $D - D'$ is principal. In this case we write $D \sim D'$.

Remark 1.5. The definition of divisor associated to a nonzero rational function f is well posed since $\mathrm{ord}_Z f \neq 0$ for a finite number of prime divisors Z (see [HAG], II, Lemma 6.1).

Remark 1.6. There exists an injective map, called *cycle map*, from $\mathrm{Div}(X)$ to $\mathrm{WDiv}(X)$ for a normal variety X over a field k:

$$\mathrm{Div}(X) \hookrightarrow \mathrm{WDiv}(X)\,, \qquad \{\{U_i\}_{i\in I}, \{f_i\}_{i\in I}\} \mapsto \sum_{Z \text{ prime}} \mathrm{ord}_Z(f_i) \cdot Z\,. \tag{1.1}$$

Basically, Cartier divisors are Weil divisors locally defined by a single equation. In general it is *not* surjective, i.e., not every Weil divisor is Cartier. It is possible to show that the surjectivity holds for smooth varieties, so by Remark 1.2 for Abelian Varieties we can identify Cartier and Weil divisors (see [HAG], II, Proposition 6.11).

There is a very straightforward way to define pullbacks of (Cartier) divisors via surjective holomorphic maps $f : X' \to X$ between smooth varieties X and X'. Given any divisor $D \in \mathrm{Div}(X)$ represented by $\{\{U_i\}_{i\in I}, \{f_i\}_{i\in I}\}$, we let $f^*(D) \in \mathrm{Div}(X')$ be given by $\{\{f^{-1}(U_i)\}_{i\in I}, \{f_i \circ f\}_{i\in I}\}$. In this way, $f : X' \to X$ induces a homomorphism of groups $f^* : \mathrm{Div}(X) \to \mathrm{Div}(X')$.

In many situations it is more convenient to work with classes of divisors modulo linear equivalence. Hence we define the *Picard group*.

Definition 1.7. Let X be a smooth variety. Then the *Picard group* is the quotient group

$$\mathrm{Pic}(X) := \mathrm{Div}(X)/\sim\,, \tag{1.2}$$

where $\sim$ denotes the linear equivalence.

It is possible to show that if X is a smooth variety then there is an isomorphism between $\mathrm{Pic}(X)$ and the group of line bundles on X modulo isomorphisms (see [HAG], II, Corollary 6.16).

We now recall some definitions given in [Yel], Chapter 2. From now on, V is a complex vector space of dimension n, Λ is a full lattice, $\mathcal{M}(V)$ is the field of meromorphic functions on V, $\pi : V \to V/\Lambda = X$ is the projection map to the complex torus X.

Definition 1.8. A function $H : V \times V \to \mathbb{C}$ is a *Hermitian form* on the complex vector space V if it is $\mathbb{C}$-linear in the first argument, and if $H(w, v) = \overline{H(v, w)}$ for all $v, w \in V$.

Remark 1.9. It is not hard to show that if $H : V \times V \to \mathbb{C}$ is a Hermitian form, its imaginary part $E : V \times V \to \mathbb{C}$ is an $\mathbb{R}$-linear alternating form, i.e., $E(w, v) = -E(v, w)$ for all $v, w \in V$, and $E(iv, iw) = E(v, w)$ for all $v, w \in V$. Conversely, any such $E : V \times V \to \mathbb{R}$ determines a Hermitian form $H : V \times V \to \mathbb{C}$ by $H(v, w) = E(iv, w) + iE(v, w)$.

Definition 1.10. A *theta function* (with respect to the lattice Λ) is a function $\theta \in \mathcal{M}(V)$ which satisfies the property that, for all $v \in V$ and $\lambda \in \Lambda$,

$$\theta(v + \lambda)/\theta(v) = e^{2\pi i(L(v,\lambda)+J(\lambda))} \tag{1.3}$$

for some map $J : \Lambda \to \mathbb{C}$ and some map $L : V \times \Lambda \to \mathbb{C}$ which is $\mathbb{C}$-linear in the first argument. Given a Hermitian form $H : V \times V \to \mathbb{C}$, we say that such a function θ is a *theta function for* H if we have $E(\lambda_1, \lambda_2) = L(\lambda_1, \lambda_2) - L(\lambda_2, \lambda_1)$ for $\lambda_1, \lambda_2 \in \Lambda$ and where E is the imaginary part of H.

Note that the set of all theta functions on V with respect to Λ forms a group under multiplication, and in fact the map from this group to the group of divisors $\mathrm{Div}(X)$ defined by sending a theta function θ to the divisor $D \in \mathrm{Div}(X)$ such that $(\theta) = \pi^* D$ is a group homomorphism which we denote by

$$\mathrm{div} : \{\text{theta functions on } V \text{ w.r.t. } \Lambda\} \to \mathrm{Div}(X)\,. \tag{1.4}$$

Its kernel is the subgroup of *trivial theta divisors*, i.e., theta divisors θ which are holomorphic and nonvanishing on V.

Proposition 1.11. *For every divisor $D \in Div(X)$, there is a unique Hermitian form H and a function $\theta \in \mathcal{M}(V)$ with $(\theta) = \pi^* D$ which is a theta function for H.*

In the proof of Proposition 1.11 it is shown that if θ is a theta function for some Hermitian form H, the imaginary part E of H must be $\mathbb{Z}$-valued on $\Lambda \times \Lambda$. This motivates the following crucial definition.

Definition 1.12. A *Riemann form* associated to a complex torus $X \cong V/\Lambda$ is a Hermitian form on V whose imaginary part is $\mathbb{Z}$-valued on $\Lambda \times \Lambda$.

Proposition 1.13. *Let $H : V \times V \to \mathbb{C}$ be the Riemann form associated to some divisor $D \in Div(X)$ via Proposition 1.11. Then any theta function $\theta \in \mathcal{M}(V)$ for H can be written as $h\theta_D$, where h is a trivial theta function, i.e., $(h) = 0 \in Div(V)$, and $\theta_D \in \mathcal{M}(V)$ is a theta function satisfying*

$$\theta_D(v + \lambda)/\theta_D(v) = e^{2\pi i\left(\frac{1}{2i}H(v,\lambda)+\frac{1}{4i}H(\lambda,\lambda)+K(\lambda)\right)} \tag{1.5}$$

where $K : \Lambda \to \mathbb{R}$ satisfies the property that

$$K(\lambda_1 + \lambda_2) - K(\lambda_1) - K(\lambda_2) \equiv \frac{1}{2}E(\lambda_1, \lambda_2) \mod \mathbb{Z}\,, \ \ \forall \lambda_1, \lambda_2 \in \Lambda\,. \tag{1.6}$$

Moreover, θ_D is unique up to a constant in $\mathbb{C}^*$.

In particular, we have:

$$\{\text{theta functions w.r.t. } \Lambda\}/\{\text{trivial theta functions}\} \xrightarrow{\sim} \mathrm{Div}(X) \to \{\text{Riemann forms w.r.t. } \Lambda\}\,. \tag{1.7}$$

1.2 Ample divisors on Complex Abelian Varieties

Let X be a Complex Abelian Variety. For any divisor $D \in \mathrm{Div}(X)$, we define the *Riemann–Roch space of D* as:

$$\mathcal{L}(D) := \{f \in \mathcal{M}(X) \,|\, (f) + D \geq 0\} \cup \{0\}\,. \tag{1.8}$$

Note that $\mathcal{L}(D)$ is a vector space. For any divisor $D \in \mathrm{Div}(X)$, let θ_0 be a theta function whose divisor is π^*D. We define $\mathcal{L}(\theta_0)$ as the set of all holomorphic functions $\theta \in \mathcal{M}(V)$ which have the same "translation functions" as θ_0 has, i.e., $\theta(v + \lambda)/\theta(v) = \theta_0(v + \lambda)/\theta_0(v)$. By convention, $0 \in \mathcal{L}(\theta_0)$. It is clear that also $\mathcal{L}(\theta_0)$ is also a vector space. In fact, it is easy to see from the definitions that we have an isomorphism $\mathcal{L}(\theta_0) \xrightarrow{\sim} \mathcal{L}(D)$ given by $\theta \mapsto \theta/\theta_0$.

We can now give the definition of *ample divisor* on a Complex Abelian Variety.

Definition 1.14. Let X be a complex abelian variety over. A divisor $D \in \mathrm{Div}(X)$ is *very ample* if there exists a basis $\{\theta_0, \dots, \theta_m\}$ of $\mathcal{L}(\theta_D)$ such that the map $\Theta : X \to \mathbb{P}^m_{\mathbb{C}}$ given by $v \mapsto (\theta_0(v) : \cdots : \theta_m(v))$ is an embedding of X into $\mathbb{P}^m_{\mathbb{C}}$, i.e., if we have the following:

(i) Θ is well-defined, i.e., we do not have $\theta_0(v) = \cdots = \theta_m(v) = 0$ for any $v \in V$.

(ii) $\Theta : X \to \mathbb{P}^m_{\mathbb{C}}$ is an injection.

(iii) The induced maps on tangent spaces $\Theta_* : T_a X \to T_{\Theta(a)}\mathbb{P}^m_{\mathbb{C}}$ are injections.

A divisor $D \in Div(X)$ is *ample* if there is an integer $n \geq 1$ such that nD is very ample.

One usually finds the following definition of ampleness in Algebraic Geometry (see [Laz1], Definition 1.2.1).

Definition 1.15. Let X be a normal variety over a field k and $\mathcal{L}$ be an invertible sheaf (or a line bundle) on X.

- We say that $\mathcal{L}$ is *very ample* if there exists an embedding $\iota : X \hookrightarrow \mathbb{P}^N$ such that $\iota^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{L}$.
- We say that $\mathcal{L}$ is *ample* if for every coherent sheaf $\mathcal{F}$ on X there exists an integer $n_0 > 0$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for every $n \geq n_0$.

Definition 1.14 and Definition 1.15 are equivalent by the following theorem due to Cartan, Serre and Grothendieck.

Theorem 1.16. *Let $\mathcal{L}$ be an invertible sheaf on a complete scheme X over a field k. The following are equivalent:*

(i) $\mathcal{L}$ is ample.

(ii) $\mathcal{L}^{\otimes m}$ is very ample for some $m > 0$.

(iii) Given $\mathcal{F}$ a coherent sheaf on X, there exists n_1 such that $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for every $i > 0$ and $n \geq n_1$ (this is known as Serre vanishing theorem).

We can now state the following fundamental theorem on Complex Abelian Varieties.

Theorem 1.17. *Let $D \in Div(X)$ be a divisor, and $H : V \times V \to \mathbb{C}$ be the corresponding Riemann form. Then H is positive definite. Conversely, if H is a positive definite Riemann form, then the divisor $D \in Div(X)$ corresponding to H is ample. More precisely, for any $n \geq 3$, the divisor $nD \in Div(X)$ is very ample.*

Hence we observe that part *(ii)* of the Fujita's Conjecture holds for Complex Abelian Varieties with lower bounds than expected

1.3 Complete linear systems and base loci

Before introducing the notion of complete linear system, we briefly recall the definition of global section of a divisor. Let X be a nonsingular projective variety. We denote by $\mathcal{O}_X(D)$ the line bundle induced by D in $\mathrm{Pic}(X)$. If $D = \{\{U_i\}_{i\in I}, \{f_i\}_{i\in I}\}$ as Cartier divisor, a *global section* s of D is a collection $s = \{\{U_i\}_{i\in I}, \{s_i\}_{i\in I}\}$ of holomorphic maps $s_i : U_i \to \mathbb{C}$ such that

$$s_i(x) = \frac{f_i}{f_j}(x)s_j(x) \qquad \text{for every } x \in U_i \cap U_j\,. \tag{1.9}$$

Observe that the notion of global section is well defined, i.e., it does not depend on the choice of the element in the class defined by D in $\mathrm{Pic}(X)$: indeed, if $D = \{\{U_i\}_{i\in I}, \{f_i\}_{i\in I}\}$ and $D' = \{\{U_i\}_{i\in I}, \{f_i'\}_{i\in I}\}$ are linearly equivalent, i.e., $D - D' = \{\{X\}, \{f\}\}$ for some nonzero rational function f (up to taking a refinement of the corresponding open covers it is possible to suppose that D and D' has the same open cover), so $f_i/f_i' = f|_{U_i}$, hence

$$s_i(x) = \frac{f_i}{f_j}(x)s_j(x) = \frac{f|_{U_i\cap U_j} \cdot f_i'}{f|_{U_i\cap U_j} \cdot f_j'}(x)s_j(x) = \frac{f_i'}{f_j'}(x)s_j(x)\,. \tag{1.10}$$

The space of global sections of D is a $\mathbb{C}$-vector space denoted by $\Gamma(X, \mathcal{O}_X(D))$, and this space can be identified with cohomology in degree zero:

$$\Gamma(X, \mathcal{O}_X(D)) \cong H^0(X, \mathcal{O}_X(D))\,. \tag{1.11}$$

See [Ram], Chapter 4, §5, for more details.

Definition 1.18. A *complete linear system* on a nonsingular projective variety X is defined as the set (maybe empty) of all effective divisors linearly equivalent to some given divisor D. It is denoted by $|D|$.

Now, we have:

Theorem 1.19 (Theorem 6.2 in [Ram])**.** *Let X be a nonsingular projective variety and $D \in Div(X)$. Then*

$$\mathcal{L}(D) \cong H^0(X, \mathcal{O}_X(D))\,. \tag{1.12}$$

Proof. Write $D = \{\{U_\alpha\}, \{f_\alpha\}\}$ as a Cartier divisor. Pick a section $s \in H^0(X, \mathcal{O}_X(D))$, i.e., $s = \{\{U_\alpha\}, \{s_\alpha\}\}$ with $s_\alpha \in \mathcal{O}(U_\alpha)$ and

$$s_\alpha(x) = \frac{f_\alpha}{f_\beta}(x)s_\beta(x) \qquad \text{for every } x \in U_\alpha \cap U_\beta\,. \tag{1.13}$$

Consider the meromorphic function $s_\alpha/f_\alpha \in \mathcal{M}(U_\alpha)$. This extends to a global meromorphic function $F \in \mathcal{M}(X)$, since we have $F|_{U_\alpha} = s_\alpha/f_\alpha = s_\beta/f_\beta$ over the intersections. Let $D = \sum m_i V_i$ as a Weil divisor, where $m_i = \mathrm{ord}_{V_i}(f_\alpha)$ for any α such that V_i and U_α intersect. Then

$$\mathrm{ord}_{V_i}(F) = \mathrm{ord}_{V_i}(s_\alpha) - \mathrm{ord}_{V_i}(f_\alpha) \geq -m_i\,, \tag{1.14}$$

which means $F \in \mathcal{L}(D)$. Thus, we have a homomorphism

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{L}(D)\,, \qquad s \mapsto F = \left(\frac{s_\alpha}{f_\alpha}\right)\,. \tag{1.15}$$

This is injective: if $F \equiv 0$, then $s_\alpha \equiv 0$ for all α, that is $s \equiv 0$. We show that it is surjective. Given $F \in \mathcal{L}(D)$, let $s_\alpha := f_\alpha \cdot F|_{U_\alpha}$. Then

$$\mathrm{ord}_{V_i}(s_\alpha) = \mathrm{ord}_{V_i}(f_\alpha) + \mathrm{ord}_{V_i}(F) \geq m_i - m_i = 0\,, \tag{1.16}$$

which implies $s_\alpha \in \mathcal{O}(U_\alpha)$. Since $s_\alpha = \frac{f_\alpha}{f_\beta} s_\beta$, this shows that $s = \{\{U_\alpha\}, \{s_\alpha\}\}$ is a global holomorphic section of $\mathcal{O}_X(D)$, and by construction $s \mapsto F$. $\square$

Remark 1.20. Let X be a nonsingular projective variety and consider $D = \{\{U_\alpha\}, \{f_\alpha\}\}$ be a divisor on X. Assume we have a global section $s = \{\{U_\alpha\}, \{s_\alpha\}\}$ in $H^0(X, \mathcal{O}_X(D))$. We define

$$\mathrm{div}(s) := \sum \mathrm{ord}_{V_i}(s_\alpha) \cdot V_i \in \mathrm{WDiv}(X)\,, \tag{1.17}$$

where $D = \sum m_i V_i$ as Weil divisor. By the proof of Theorem 1.19 we have

$$\mathrm{div}(s) = \sum (m_i + \mathrm{ord}_{V_i}(F)) \cdot V_i = D + (F)\,, \tag{1.18}$$

where $(F) = \sum \mathrm{ord}_{V_i}(F) \cdot V_i$ is the divisor associated to the meromorphic function $F \in \mathcal{M}(X)$ constructed as in the proof of Theorem 1.19. Since we have $m_i + \mathrm{ord}_{V_i}(F) \geq 0$, we see that $D' := \mathrm{div}(s)$ is an effective divisor. Moreover, we have $D' - D = (F)$, which means that D' and D are linearly equivalent. Thus, $\mathrm{div}(s) \in |D|$, and we have a map

$$H^0(X, \mathcal{O}_X(D)) \to |D|\,, \qquad s \mapsto \mathrm{div}(s)\,. \tag{1.19}$$

Notice that $\mathrm{div}(s) = \mathrm{div}(\lambda s)$ for any non-zero scalar. Moreover, it is an easy exercise to show that any divisor in $|D|$ arises as the divisor of some section of D (see [HAG], II, Proposition 7.7 for more details). Therefore

$$|D| \cong \mathbb{P}(H^0(X, \mathcal{O}_X(D)))\,. \tag{1.20}$$

Hence, if X is a Complex Abelian Variety, we have:

$$|D| \cong \mathbb{P}(\mathcal{L}(D)) \cong \mathbb{P}(\mathcal{L}(\theta_D)) . \tag{1.21}$$

Definition 1.21. Let X be a nonsingular projective variety and D a divisor on X. The *support* of D, denoted by $\operatorname{Supp} D$, is the union of the prime divisors of D.

We can finally give the definition of base point of a complete linear system.

Definition 1.22. Let X be a nonsingular projective variety and consider a divisor $D \in Div(X)$. A point $p \in X$ is a *base point* for the complete linear system $|D|$ if $p \in \operatorname{Supp} D$ for all $D' \in |D|$, i.e., $s(x) = 0$ for every $s \in H^0(X, \mathcal{O}_X(D))$. The *base locus* of $|D|$ is given by:

$$\operatorname{Bs}|D| := \{x \in X \mid s(x) = 0 \text{ for every } s \in H^0(X, \mathcal{O}_X(D))\} . \tag{1.22}$$

We say that $|D|$ is *basepoint free* if $\operatorname{Bs}|D| = \emptyset$

Hence, given a divisor $D \in \operatorname{Div}(X)$ for a nonsingular projective variety X, it is possible to define the *map induced by the linear system* $|D|$, denoted by $\phi_{|D|}$, as follows:

$$\phi_{|D|} : X \dashrightarrow \mathbb{P}(|D|^\vee) , \qquad x \mapsto (s_0(x) : \cdots : s_N(x)) , \tag{1.23}$$

where $\{s_0, \ldots, s_N\}$ is a basis of $H^0(X, \mathcal{O}_X(D))$. Clearly the map is not defined in the base locus $\operatorname{Bs}|D|$ and it is a morphism in $X - \operatorname{Bs}|D|$, so if $|D|$ is basepoint free the map $\phi_{|D|}$ is a morphism. The map is well-defined in $X - \operatorname{Bs}|D|$: if $D = \{\{U_\alpha\}, \{f_\alpha\}\}$, $s_i = \{\{U_\alpha\}, \{s_{i,\alpha}\}\}$, $s_j = \{\{U_\alpha\}, \{s_{j,\alpha}\}\}$, where $X = \cup_\alpha U_\alpha$, for $x \in U_\alpha \cap U_\beta$ the vectors $(s_{0,\alpha}(x) : \cdots : s_{N,\alpha}(x))$ and $(s_{0,\beta}(x) : \cdots : s_{N,\beta}(x))$ differ by multiplication by $f_\alpha/f_\beta(x) \neq 0$, hence they are the same point in the projective space.

If X is a Complex Abelian Variety, if $\{\theta_0, \ldots, \theta_N\}$ is a basis of $\mathcal{L}(\theta_D)$, the map induced by the linear system $|D|$ is:

$$\phi_{|D|} : X \dashrightarrow \mathbb{P}((\mathcal{L}(\theta_D))^\vee) , \qquad v \mapsto (\theta_0(v) : \cdots : \theta_N(v)) , \tag{1.24}$$

and the base locus of $|D|$ is the set of points where all the θ_i vanish. Basically, if X is a Complex Abelian Variety and $D \in \operatorname{Div}(X)$, we say that $|D|$ is basepoint free if (i) of Definition 1.14 holds.

2 Some technical results

We present some results we need for the proof of the main theorem.

Lemma 2.1. *Let $f : Y \to X$ be a finite morphism of complete schemes. If $\mathcal{L}$ is an invertible ample sheaf (or line bundle) on X, then $f^*\mathcal{L}$ is ample.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on Y. Since f is finite we have:

$$H^i(Y, \mathcal{F} \otimes f^*\mathcal{L}^{\otimes n}) = H^i(X, f_*\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0 \qquad \text{for every } i \text{ and } n \gg 0\,, \tag{2.1}$$

where the first equality holds since f is finite, hence proper, and proper morphisms preserve coherence, while the second is obtained using Serre vanishing theorem. □

We now recall a special case of the *Projection formula* (see [HAG], Appendix A, 1, A4).

Theorem 2.2 (Projection formula)**.** *Let $f : X \to X'$ be a proper surjective morphism of nonsingular projective varieties over $\mathbb{C}$. Let $C \subset X$ be a curve on X and $H \in Div(X')$ be a divisor on X'. Then:*

$$C \cdot f^*H = f_*(C) \cdot H\,, \tag{2.2}$$

where, if $dim\, f(C) < dim\, C$ we set $f_*(C) = 0$, otherwise, if $dim\, f(C) = dim\, C$ we set:

$$f_*(C) := [\mathcal{M}(C) : \mathcal{M}(f(C))] \cdot \overline{f(C)}\,. \tag{2.3}$$

Theorem 2.3. *Suppose that $f : X \to Y$ is a projective morphism, and $\mathcal{F}$ is a coherent sheaf on X, flat over Y. Suppose that Y is locally Noetherian. Then $\sum(-1)^i h^i(X_y, \mathcal{F}_{|y})$ is a locally constant function of $y \in Y$, i.e., the Euler characteristic of $\mathcal{F}$ is constant in the fibers.*

Proof. See [Vak], Theorem 1.1. □

Corollary 2.4. *An invertible sheaf $\mathcal{L}$ on a flat projective family of connected nonsingular curves has locally constant degree on the fibers.*

Proof. An invertible sheaf $\mathcal{L}$ on a flat family of curves is always flat, since it is locally isomorphic to the structure sheaf. Hence by Theorem 2.3 we have that $\chi(\mathcal{L}_y)$ is constant. From the Riemann–Roch formula $\chi(\mathcal{L}_y) = \deg(\mathcal{L}_y) - g(X_y) + 1$. Using the local constancy of $\chi(\mathcal{L}_y)$ the result follows. □

Theorem 2.5 (Seesaw Theorem)**.** *Let X and Y be normal varieties. Suppose that X is complete. Let L and H be two line bundles on $X \times Y$. The set $\{y \in Y$ s.t. $L|_{X\times\{y\}} \cong M|_{X\times\{y\}}\}$ is Zariski-closed. Moreover, if for all closed $y \in Y$ we have $L_y \cong M_y$, i.e., $L_{|X\times\{y\}} \cong M_{|X\times\{y\}}$, then there exists a line bundle N on Y such that $L \cong M \otimes p^*N$, where $p = pr_Y : X \times Y \to Y$ is the projection onto Y. If in addition we have $L_x = M_x$ for some $x \in X$, then $L \cong M$.*

Proof. We do not prove that $\{y \in Y$ s.t. $L|_{X\times\{y\}} \cong M|_{X\times\{y\}}\}$ is Zariski-closed (see [Mum], Corollary 6). Now, since $L_y \otimes M_y^{-1}$ is the trivial bundle and X_y is complete, the space of sections $H^0(X_y, L_y \otimes M_y^{-1})$ is isomorphic to $k(y)$, the residue field of y. This implies that $p_*(L \otimes M^{-1})$ is locally free of rank one, hence a line bundle (see [HAG], III, §12). We shall prove that the natural map

$$\alpha : p^*p_*(L \otimes M^{-1}) \to L \otimes M^{-1} \tag{2.4}$$

is an isomorphism. If we restrict to a fibre we find the map

$$\mathcal{O}_{X_y} \otimes \Gamma(X_y, \mathcal{O}_{X_y}) \to \mathcal{O}_{X_y} \tag{2.5}$$

which is an isomorphism. By Nakayama's lemma, this implies that α is surjective and by comparing ranks we conclude that it is an isomorphism, hence $L \cong M \otimes p^*N$. Over $\{x\} \times Y$ this gives $L_x \cong M_x \otimes (\mathrm{pr}_Y^* N)_x$. Therefore $(\mathrm{pr}_Y^* N)_x$ is trivial, and this implies that N is trivial. □

Theorem 2.6 (Theorem of the cube)**.** *Let X, Y be complete varieties, Z any variety and x_0, y_0 and z_0 base points on X, Y and Z respectively. If L is any line bundle on $X \times Y \times Z$ whose restrictions to each of $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ are trivial, then L is trivial.*

Proof. We follow [Ort], Theorem 12, and we give a proof over $\mathbb{C}$ using the exponential sequence, at least in the case Z is also complete. For simplicity, assume none of X, Y and Z have torsion in their cohomology. Then the Künneth theorem tells us that the natural map

$$H^*(X, \mathbb{Z}) \otimes H^*(Y, \mathbb{Z}) \otimes H^*(Z, \mathbb{Z}) \xrightarrow{p_1^* \cup p_2^* \cup p_3^*} H^*(X \times Y \times Z, \mathbb{Z}) \tag{2.6}$$

is an isomorphism, where p_i is the projection to the i-th factor (we let p_{ij} be the projection to the i-th and j-th factors). Let ι_i be the inclusion of the i-th factor and ι_{ij} the inclusion of the i-th and j-th factors using the basepoints. Concretely,

$$\iota_1 : X \times y_0 \times z_0 \to X \times Y \times Z\,, \qquad \iota_{12} : X \times Y \times z_0 \to X \times Y \times Z\,. \tag{2.7}$$

If you think about the isomorphism (2.6) in degree 2, for any class $\alpha \in H^2(X \times Y \times Z, \mathbb{Z})$ we have

$$\alpha = \alpha_{12} + \alpha_{13} + \alpha_{23} - \alpha_1 - \alpha_2 - \alpha_3\,, \tag{2.8}$$

where α_{ij} (respectively α_i) is α inserted in the i-th and j-th slots (respectively $p_i^*\alpha$ inserted in the i-th slot) via the Künneth formula, so $\alpha_{12} = \iota_{12}^*\alpha \otimes 1$, $\alpha_1 = \iota_1^*\alpha \otimes 1 \otimes 1$. In particular, this means that if $\iota_{ij}^*\alpha = 0$ for all i, j, then $\alpha = 0$.

The long exact sequence associated to the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \tag{2.9}$$

gives us an exact sequence

$$H^1(X \times Y \times Z, \mathcal{O}) \xrightarrow{\exp} H^1(X \times Y \times Z, \mathcal{O}^*) \xrightarrow{c_1} H^2(X \times Y \times Z, \mathbb{Z})\,. \tag{2.10}$$

Given a line bundle L on $X \times Y \times Z$ thought of as an element of the middle group, $\iota_{ij}^* c_1(L) = c_1(\iota_{ij}^* L) = 0$ for all i, j, so $c_1(L) = 0$ and $L = \exp(A)$ for some $A \in H^1(X \times Y \times Z, \mathcal{O})$. But

$$H^1(X, \mathcal{O}) \oplus H^1(Y, \mathcal{O}) \oplus H^1(Z, \mathcal{O}) \xrightarrow{p_1^*+p_2^*+p_3^*} H^1(X \times Y \times Z, \mathcal{O}) \tag{2.11}$$

is an isomorphism, and the hypothesis imply that $\iota_i^*\exp(A) = \exp(\iota_i^* A) = 0$ for each i. Thus,

$$L = \exp\left(\sum_i \iota_i^* A\right) = 0\,, \tag{2.12}$$

as we wanted. □

Remark 2.7. Note that the proof of Theorem 2.6 that we have given is good if X, Y and Z are Complex Abelian Varieties, since these are complete varieties by definition and their cohomology groups have no torsion (see [Fil], Theorem 3.3).

Corollary 2.8. *Let X be any variety, Y an Abelian Variety, and $f, g, h : X \to Y$ morphisms. Then for all line bundles $L \in Pic(Y)$ we have:*

$$(f+g+h)^*L \cong (f+g)^*L \otimes (f+h)^*L \otimes (g+h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}\,. \tag{2.13}$$

Proof. Let $p_i : Y \times Y \times Y \to Y$ be the projection onto the i-th factor, put $m_{ij} := p_i + p_j : Y \times Y \times Y \to Y$ and $m := p_1 + p_2 + p_3 : Y \times Y \times Y \to Y$.

Consider the line bundle

$$M := m^*L \otimes m_{12}^*L^{-1} \otimes m_{13}^*L^{-1} \otimes m_{23}^*L^{-1} \otimes p_1^*L \otimes p_2^*L \otimes p_3^*L \quad (2.14)$$

on $Y \times Y \times Y$. If $q : Y \times Y \to Y \times Y \times Y$ is the map given by $q(y, y') = (0, y, y')$, we have

$$q^*M = n^*L \otimes q_1^*L^{-1} \otimes q_2^*L^{-1} \otimes n^*L^{-1} \otimes 0^*L \otimes q_1^*L \otimes q_2^*L \qquad (2.15)$$

where $0, q_1, q_2, n : Y \times Y \to Y$ are respectively the 0 map, the projections, and the addition. Therefore q^*M is trivial. By symmetry, M is trivial on $Y \times \{0\} \times Y$ and $Y \times Y \times \{0\}$ too. By Theorem 2.6, M must be trivial on $Y \times Y \times Y$. Pulling back M by the map $(f, g, h) : X \to Y \times Y \times Y$, the result follows. □

Corollary 2.9 (Theorem of the square)**.** *Let X be a Complex Abelian Variety. Then for all line bundles L and $x, y \in X$ we have:*

$$t_{x+y}^*L \otimes L \cong t_x^*L \otimes t_y^*L \,. \qquad (2.16)$$

Therefore if we define the map

$$\phi_L(x) := \textit{isom. class of } t_x^*L \otimes L^{-1} \textit{ in } Pic(X) \,, \qquad (2.17)$$

then ϕ_L is a homomorphism from X to $Pic(X)$.

Proof. Apply Corollary 2.8 with $X = Y$, f and g constant maps with images x, y respectively, and $h =$ identity. □

In terms of divisors, Corollary 2.9 asserts that for any divisor D on X, and $x, y \in X$,

$$t_{x+y}^*D + D \sim t_x^*D + t_y^*D \,, \qquad (2.18)$$

where $\sim$ denotes the linear equivalence. From now on, we will always keep the notation ϕ_L for this very important map. Note that:

1. $\phi_{L_1 \otimes L_2} = \phi_{L_1} + \phi_{L_2}$, where $+$ stands for the group law induced by $\otimes$ in $\mathrm{Pic}(X)$.

2. $\phi_{t_x^*L} = \phi_L$.

Definition 2.10. Let L be a line bundle on a Complex Abelian Variety X. Then we set:

$$K(L) := \mathrm{Ker}(\phi_L) = \{x \in X \,|\, t_x^*L \cong L\} \,. \qquad (2.19)$$

Proposition 2.11. *Let L be a line bundle on a Complex Abelian Variety X. Then $K(L)$ is a Zariski-closed subgroup of X.*

Proof. Apply the Seesaw Theorem to the line bundle $m^*L \otimes p_2^*L^{-1}$ on $X \times X$, where $m : X \times X \to X$ denotes the addition. It follows that the set of $x \in X$ such that $m^*L \otimes p_2^*L^{-1}$ is trivial on $\{x\} \times X$ is Zariski closed. But $m^*L \otimes p_2^*L^{-1}|_{\{x\} \times X} \cong t_x^*L \otimes L^{-1}$, so this set is $K(L)$. □

3 Basepoint freeness theorem

We now state the main theorem of the presentation.

Theorem 3.1. *Let X be a Complex Abelian Variety and let D be an effective divisor on X and $L = L(D)$ the associated line bundle. The following conditions are equivalent:*

(i) The complete linear system $|2D|$ has no base points, and defines a finite morphism $\phi_{|2D|} : X \to \mathbb{P}^N$.

(ii) L is ample on X, i.e., D is ample on X.

(iii) $K(L)$ is finite.

(iv) The subgroup $H := \{x \in X \mid t_x^*(D) = D\}$ of X is finite (equality of divisors, not divisor classes).

Remark 3.2. The result is in fact true for abelian varieties over an algebraically closed field k: the only difference in the proof is that the theorem of the cube has a more complicated proof than the one given before.

Proof. "$(i) \implies (ii)$" It is a consequence of Lemma 2.1, since by assumption $\phi_{|2D|}$ is finite and clearly $\mathcal{O}_{\mathbb{P}^N}(1)$ is ample.

"$(ii) \implies (iii)$" If $K(L)$ is not finite, let Y be the connected component of $K(L)$ containing 0, which obviously lies on $K(L)$. We show that Y is an abelian variety of positive dimension. Indeed, it is a group variety since X is a group variety by Definition 1.1, and it is closed by Proposition 2.11. Furthermore, for every variety Z the projection $Y \times Z \to Z$ is closed since it is the composition of the closed maps $Y \times Z \hookrightarrow X \times Z$ and $X \times Z \to Z$, where the first map is the closed immersion of $Y \times Z$ in $X \times Z$ and the second map is the projection of $X \times Z$ on Z, and the latter is a closed map since X is complete. Thus Y is complete, hence it is an abelian variety by Definition 1.1. Now, by assumption L is ample on X, i.e., for $n \gg 0$ the map induced by the linear system $|nL|$ is an embedding, hence $\phi_{|nL|_{|Y}} \equiv \phi_{|nL_Y|}$ is an embedding of Y into some projective space, where L_Y is the restriction of L to Y. That means that L_Y is ample on Y. Moreover, since $Y \subseteq K(L) = \ker(\phi_L)$, we have $t_y^*(L_Y) \cong L_Y$ for all $y \in Y$. Let $m_Y : Y \times Y \to Y$ be the addition on Y and $p_i : Y \times Y \to Y$ be the projections for $i = 1, 2$.

Claim 3.3. The line bundle $m_Y^*(L_Y) \otimes p_1^*(L_Y^{-1}) \otimes p_2^*(L_Y^{-1})$ is trivial on $Y \times Y$.

Proof Claim 3.3. We apply Seesaw Theorem. Consider the line bundle $m_Y^* L_Y \otimes p_2^* L_Y^{-1}$ on $Y \times Y$. Clearly for every $y \in Y$ we have:

$$(m_Y^* L_Y \otimes p_2^* L_Y^{-1})_{\{y\} \times Y} \cong (p_1^* L_Y)_{\{y\} \times Y} \tag{3.1}$$

since the first line bundle is isomorphic to $t_y^* L_Y \otimes L_Y^{-1}$, which is isomorphic by definition of Y to $L_Y \otimes L_Y^{-1}$, hence trivial, and the second line bundle is obviously trivial. Hence by the Seesaw Theorem we get for some line bundle M on Y:

$$m_Y^* L_Y \otimes p_2^* L_Y^{-1} \cong p_1^* M \,. \tag{3.2}$$

To conclude we want to show that $M = L_Y$. Observe that:

$$(m_Y^* L_Y \otimes p_2^* L_Y^{-1})_{|Y \times \{y\}} \cong (p_1^* L_Y)_{|Y \times \{y\}} \tag{3.3}$$

since both are isomorphic to L_Y, hence by the last part of Seesaw Theorem we have $M = L_Y$.

Now from $m_Y^* L_Y \otimes p_2^* L_Y^{-1} \cong p_1^* L_Y$ we have that $m_Y^* L_Y \otimes p_1^* L_Y^{-1} \otimes p_2^* L_Y^{-1}$ is trivial on $Y \times Y$. □

Consider now the composition

$$Y \xrightarrow{g:=(\mathrm{id},-\mathrm{id})} Y \times Y \xrightarrow{m_Y} Y \,. \tag{3.4}$$

Clearly, also $m_Y^* L_Y^{-1} \otimes p_1^* L_Y \otimes p_2^* L_Y$ is trivial, hence

$$\begin{aligned} g^* \mathcal{O}_{Y \times Y} &\cong g^*(m_Y^* L_Y^{-1} \otimes p_1^* L_Y \otimes p_2^* L_Y) \\ &\cong (m_Y \circ g)^* L_Y^{-1} \otimes (p_1 \circ g)^* L_Y \otimes (p_2 \circ g)^* L_Y \\ &\cong (-\mathrm{id})^* L_Y \otimes L_Y \,, \end{aligned} \tag{3.5}$$

and this is trivial on Y, being g^* a group homomorphism. But we have seen that L_Y is ample on Y, and since $(-\mathrm{id})$ is an automorphism of Y, $L_Y \otimes (-\mathrm{id})^*(L_Y) \cong g^* \mathcal{O}_{Y \times Y}$ is ample. So $L_Y \otimes (-\mathrm{id})^*(L_Y)$ is both ample and trivial, and this is a contradiction since $\dim Y > 0$.

"$(iii) \implies (iv)$" It is trivial since $K(L) \supset H$.

"$(iv) \implies (i)$" The linear system $|2D|$ contains the divisors $t_x^*(D) + t_{-x}^*(D)$ by (2.18). Since we have observed in Remark 1.2 that the translation map is bijective, for any $u \in X$ we can find an $x \in X$ such that $u \pm x \notin \operatorname{Supp} D$, and this means that $u \notin \operatorname{Supp}(t_x^*(D) + t_{-x}^*(D))$. Thus the linear system $|2D|$ has no base points, and defines a morphism $\phi_{|2D|} : X \to \mathbb{P}^N$. We now show that $\phi_{|2D|}$ is a finite morphism. Suppose that it is not finite: thus we can find an irreducible curve C such that $\phi_{|2D|}(C) = p$ is a point in $\mathbb{P}^N$. It follows that for all $E \in |2D|$, either E contains C or is disjoint from C. Indeed, suppose that C is not contained in E. Recall that $\phi_{|2D|}^* H = 2D$, where H is the divisor defined by a hyperplane in $\mathbb{P}^N$. Then we have:

$$E \cdot C = 2D \cdot C = \phi_{|2D|}^* H \cdot C = H \cdot \phi_{|2D|_*} C \,, \tag{3.6}$$

where the last equality comes form the projection formula (PF), and $H \cdot \phi_{|2D|_*}(C) = 0$ since $\phi_{|2D|}(C) = p$. In particular, for almost all $x \in X$, we have that C and $t_x^*(D)+t_{-x}^*(D)$ are disjoint. Obviously it is not possible that for every $E \in |2D|$ the curve C is contained in E, since this would imply that $C \subset \mathrm{Bs}\,|2D|$, but $\mathrm{Bs}\,|2D| = \emptyset$. Moreover, $\mathrm{Supp}(2D) \cap C = \emptyset$ since $2D = \phi_{|2D|}^* H$ and $\phi_{|2D|}(C)$ is a point. Now note the following general fact.

Lemma 3.4. *Let X be a Complex Abelian Variety. If C is a curve on X and E is an irreducible divisor on X such that $C \cap E = \emptyset$, then E is invariant under translation by $x_1 - x_2$, for all $x_i \in C$.*

Proof Lemma 3.4. If L is the line bundle given by the class of E in $\mathrm{Pic}(X)$, then L is trivial on C since C and E are disjoint. We want to show that t_x^*L, restricted to C, has degree 0 for all $x \in X$. Indeed, up to normalization we can suppose that C is nonsingular. Let $m_C : C \times X \to X$ be the addition and consider the line bundle m_C^*L on $C \times X$. So for any $x \in X$:

$$\chi((t_x^*L)_{|C}) = \chi((m_C^*L)_{|C\times\{x\}}) = \chi((m_C^*L)_{|C\times\{0\}}) = \chi(\mathcal{O}_C)\,, \tag{3.7}$$

since by Theorem 2.3 and Corollary 2.4 the Euler characteristic is constant in the fibers of m_C. Hence $\deg((t_x^*L)_{|C}) = \deg(\mathcal{O}_C) = 0$. But then using the projection formula we have that $t_x(C)$ and E can never intersect in a non-empty finite set of points, since this would imply that $t_x^*(L)_{|C}$ had positive degree. Thus for all $x \in X$, either $t_x(C)$ and E are disjoint, or $t_x(C) \subset E$. Let $x_1, x_2 \in C$, $y \in E$. Then $t_{y-x_2}(C)$ and E meet at y. Therefore $t_{y-x_2}(C) \subset E$, hence $y - x_2 + x_1 \in E$. This proves the lemma. □

If $D = \sum n_i D_i$, with D_i irreducible, then by Lemma 3.4 each D_i is invariant under translation by all points $x_1 - x_2$, $x_i \in C$. This contradicts (iv), hence we have proved Theorem 3.1. □

Example 3.5. Let X be a Riemann surface of genus 1, i.e., an elliptic curve over $\mathbb{C}$. Consider D a divisor on X with $\deg D > 0$. Since $g = 1$ and $l(K-D) = 0$, being $\deg(K-D) = \deg K - \deg D = 2g - 2 - \deg D < 0$, Riemann–Roch implies

$$l(D) = \deg D\,. \tag{3.8}$$

If we denote by $D(p)$ the coefficient of a point $p \in X$ in $D = \sum D(p_i)p_i$, for every $f \in \mathcal{L}(D)$ and for every point $p \in X$ we have $D(p) + \mathrm{ord}_p(f) \geq 0$. So by Theorem 1.19 we have that p is a basepoint of $\mathcal{L}(D) = |D|$ if and only if for every $f \in \mathcal{L}(D)$ one has $D(p) + \mathrm{ord}_p(f) \geq 1$. Since f is already in $\mathcal{L}(D)$, this condition on f exactly says that $f \in \mathcal{L}(D - p)$. Recall also the following result.

Lemma 3.6 (Lemma 3.15 in [Mir], Ch.V)**.** *Let X be a Riemann surface, let D be a divisor on X, and let p be a point of X. Then either $\mathcal{L}(D-p) = \mathcal{L}(D)$ or $\mathcal{L}(D-p)$ has codimension one in $\mathcal{L}(D)$.*

Proof. Choose a local coordinate z centered at p, and let $n = -D(p)$. Then every function f in $\mathcal{L}(D)$ has a Laurent series at p of the form cz^n+ higher terms. Define a map $\alpha : \mathcal{L}(D) \to \mathbb{C}$ by sending f to the coefficient of the z^n term in its Laurent series. Clearly α is a linear map and the kernel of α is exactly $\mathcal{L}(D-p)$. If α is the identically zero map, then $\mathcal{L}(D-p) = \mathcal{L}(D)$. Otherwise α is onto, and so $\mathcal{L}(D-p)$ has codimension one in $\mathcal{L}(D)$. □

Hence we have the following proposition.

Proposition 3.7 (Proposition 4.9 in [Mir], Ch.V)**.** *Let D be a divisor on a compact Riemann surface X. Then a point $p \in X$ is a basepoint of the complete linear system $|D|$ if and only if $\dim \mathcal{L}(D-p) = \dim \mathcal{L}(D)$. Hence $|D|$ is basepoint free if and only if for every point $p \in X$ we have $\dim \mathcal{L}(D-p) = \dim \mathcal{L}(D) - 1$.*

So take a point $p \in X$, where X is a genus 1 Riemann surface. From 3.8 we have $l(p) = 1$, and obviously $\mathbb{C} \cong \{\text{constant functions on}\, X\} \subseteq \mathcal{L}(p)$, hence $\mathcal{L}(p) \cong \mathbb{C}$. Now, $\mathcal{L}(p-p) = \mathcal{L}(0) \cong \mathbb{C}$ (here 0 is *not* the distinguished point of the definition of elliptic curve) hence Proposition 3.7 implies that p is a basepoint for $|p|$. Take now $q \neq p$ in X. Clearly $\mathcal{L}(p-q) \subset \mathcal{L}(p)$. If $\mathcal{L}(p-q) = \mathcal{L}(p)$, then $\mathcal{L}(p-q) \cong \mathbb{C}$, but if f is a nonzero constant function it is not true that $(f)+p-q \geq 0$, so $f \notin \mathcal{L}(p-q)$, hence $\mathcal{L}(p-q) = \{0\}$ and $l(p-q) = l(p) - 1$. By Proposition 3.7 we have that every $q \neq p$ is outside $\mathrm{Bs}\,|p|$.

Consider now the divisor $2p$. By (3.8) we have

$$l(2p) = 2\,, \tag{3.9}$$

hence $\mathcal{L}(p)$ is a $\mathbb{C}$-vector space with basis $\{1, f\}$, where f is a meromorphic function with a pole of order 2 at p. A point q is a basepoint for the complete linear system $|2p|$ if and only if $\mathcal{L}(2p-q) = \mathcal{L}(2p)$, but by (3.8) we have that $l(2p) = 2$ and $l(2p-q) = 1$ for every $q \in X$, hence by Proposition 3.7 we conclude that $|2p|$ is basepoint free. The divisor $2p$ is not very ample, otherwise we would have an injective map $X \hookrightarrow \mathbb{P}^1$, but the only possible images of $[1 : f]$ are constant or the entire Riemann sphere, since X is compact and connected. But X has nontrivial fundamental group, so this is a contradiction, hence $2p$ is not very ample. Another way to see this is using the following result.

Proposition 3.8 (Proposition 4.20 in [Mir], Ch.V). *Let X be a compact Riemann surface, and let D be a divisor on X whose linear system $|D|$ has no basepoints. Then ϕ_D, the map induced by $|D|$, is a $1-1$ holomorphic map and an isomorphism onto its image if and only if for every p and q in X we have $dim\,\mathcal{L}(D-p-q) = dim\,\mathcal{L}(D) - 2$.*

Since $\mathcal{L}(2p-p-p) = \mathcal{L}(0)$ and $l(0) = 1$ while $l(2p) = 2$, by Proposition 3.8 we conclude that $2p$ is not very ample.

Recall the *Riemann-Hurwitz's formula.*

Theorem 3.9 (Theorem 4.16 in [Mir], Ch.II). *Let $F : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then*

$$2g(X) - 2 = deg(F)(2g(Y) - 2) + \sum_{p \in X} (mult_p(F) - 1)\,. \tag{3.10}$$

In our case, this implies that $\phi_{|2p|}$ has p and three other points in the ramification locus.

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